Tight Regret Bounds for Model-Based Reinforcement Learning with Greedy Policies

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Abstract

State-of-the-art efficient model-based Reinforcement Learning (RL) algorithms typically act by iteratively solving empirical models, i.e., by performing *full-planning* on Markov Decision Processes (MDPs) built by the gathered experience. In this paper, we focus on model-based RL in the finite-state finite-horizon undiscounted MDP setting and establish that exploring with *greedy policies* – act by *1-step planning* – can achieve tight minimax performance in terms of regret, $\tilde{\mathcal{O}}(\sqrt{HSAT})$. Thus, full-planning in model-based RL can be avoided altogether without any performance degradation, and, by doing so, the computational complexity decreases by a factor of S. The results are based on a novel analysis of real-time dynamic programming, then extended to model-based RL. Specifically, we generalize existing algorithms that perform full-planning to act by 1-step planning. For these generalizations, we prove regret bounds with the same rate as their full-planning counterparts.

1 Introduction

Reinforcement learning (RL) [Sutton and Barto, 2018] is a field of machine learning that tackles the problem of learning how to act in an *unknown* dynamic environment. An agent interacts with the environment, and receives feedback on its actions in the form of a state-dependent reward signal. Using this experience, the agent's goal is then to find a policy that maximizes the long-term reward.

There are two main approaches for learning such a policy: model-based and model-free. The model-based approach estimates the system's model and uses it to assess the long-term effects of actions via *full-planning* (e.g., Jaksch et al. 2010). Model-based RL algorithms usually enjoy good performance guarantees in terms of the regret – the difference between the sum of rewards gained by playing an optimal policy and the sum of rewards that the agent accumulates [Jaksch et al., 2010, Bartlett and Tewari, 2009]. Nevertheless, model-based algorithms suffer from high space and computation complexity. The former is caused by the need for storing a model. The latter is due to the frequent full-planning, which requires a full solution of the estimated model. Alternatively, model-free RL algorithms directly estimate quantities that take into account the long-term effect of an action, thus, avoiding model estimation and planning operations altogether [Jin et al., 2018]. These algorithms usually enjoy better computational and space complexity, but seem to have worse performance guarantees.

In many applications, the high computational complexity of model-based RL makes them infeasible. Thus, practical model-based approaches alleviate this computational burden by using *short-term planning* e.g., Dyna [Sutton, 1991], instead of full-planning. To the best of our knowledge, there are no regret guarantees for such algorithms, even in the tabular setting. This raises the following question: *Can a model-based approach coupled with short-term planning enjoy the favorable performance of model-based RL?*

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Algorithm	Regret	Time Complexity	Space Complexity
UCRL2[2] [Jaksch et al., 2010]	$\tilde{O}(\sqrt{H^2S^2AT})$	$\tilde{\mathcal{O}}(\mathcal{N}SAH)$	$\tilde{\mathcal{O}}(HS+\mathcal{N}SA)$
UCBVI [Azar et al., 2017]	$\tilde{O}(\sqrt{HSAT}+\sqrt{H^2T})$	$\tilde{\mathcal{O}}(\mathcal{N}SAH)$	$\tilde{\mathcal{O}}(HS+\mathcal{N}SA)$
EULER [Zanette and Brunskill, 2019]	$\tilde{\mathcal{O}}(\sqrt{HSAT})$	$\tilde{\mathcal{O}}(\mathcal{N}SAH)$	$\tilde{\mathcal{O}}(HS+\mathcal{N}SA)$
UCRL2-GP	$\tilde{O}(\sqrt{H^2S^2AT})$	$\tilde{\mathcal{O}}(\mathcal{N}AH)$	$\tilde{\mathcal{O}}(HS+\mathcal{N}SA)$
EULER-GP	$\tilde{\mathcal{O}}(\sqrt{HSAT})$	$\tilde{\mathcal{O}}(\mathcal{N}AH)$	$\tilde{\mathcal{O}}(HS+\mathcal{N}SA)$
Q-v2 [Jin et al., 2018]	$\tilde{O}(\sqrt{H^3SAT}$	$\tilde{\mathcal{O}}(AH)$	$\tilde{\mathcal{O}}(HSA)$
Lower bounds	$\Omega\left(\sqrt{HSAT}\right)$	–	–

Table 1: Comparison of our bounds with several state-of-the-art bounds for RL in tabular finite-horizon MDPs. The time complexity of the algorithms is per episode; S and A are the sizes of the state and action sets, respectively; H is the horizon of the MDP; T is the total number of samples that the algorithm gathers; $\mathcal{N} \leq S$ is the maximum number of non-zero transition probabilities across the entire state-action pairs. The algorithms proposed in this paper are highlighted in gray.

In this work, we show that model-based algorithms that use 1-step planning can achieve the same performance as algorithms that perform full-planning, thus, answering affirmatively to the above question. To this end, we study Real-Time Dynamic-Programming (RTDP) [Barto et al., 1995] that finds the optimal policy of a *known* model by acting greedily based on 1-step planning, and establish new and sharper finite sample guarantees. We demonstrate how the new analysis of RTDP can be incorporated into two model-based RL algorithms, and prove that the regret of the resulting algorithms remains unchanged, while their computational complexity drastically decreases. As Table 1 shows, this reduces the computational complexity of model-based RL methods by a factor of S.

The contributions of our paper are as follows: we first prove regret bounds for RTDP when the model is known. To do so, we establish concentration results on Decreasing Bounded Processes, which are of independent interest. We then show that the regret bound translates into a Uniform Probably Approximately Correct (PAC) [Dann et al., 2017] bound for RTDP that greatly improves existing PAC results [Strehl et al., 2006]. Next, we move to the learning problem, where the model is unknown. Based on the analysis developed for RTDP we adapt UCRL2 [Jaksch et al., 2010] and EULER [Zanette and Brunskill, 2019], both act by full-planning, to UCRL2 with Greedy Policies (UCRL2-GP) and EULER with Greedy Policies (EULER-GP); model-based algorithms that act by 1-step planning. The adapted versions are shown to preserve the performance guarantees, while improve in terms of computational complexity.

2 Notations and Definitions

We consider finite-horizon MDPs with time-independent dynamics [Bertsekas and Tsitsiklis, 1996]. A finite-horizon MDP is defined by the tuple $\mathcal{M} = (\mathcal{S}, \mathcal{A}, R, p, H)$, where $\mathcal{S}$ and $\mathcal{A}$ are the state and action spaces with cardinalities S and A, respectively. The immediate reward for taking an action a at state s is a random variable $R(s,a) \in [0,1]$ with expectation $\mathbb{E}R(s,a) = r(s,a)$. The transition probability is $p(s' \mid s,a)$, the probability of transitioning to state s' upon taking action a at state s. Furthermore, $\mathcal{N} := \max_{s,a} |\{s' : p(s' \mid s,a) > 0\}|$ is the maximum number of non-zero transition probabilities across the entire state-action pairs. If this number is unknown to the designer of the algorithm in advanced, then we set $\mathcal{N} = S$. The initial state in each episode is arbitrarily chosen and $H \in \mathbb{N}$ is the *horizon*, i.e., the number of time-steps in each episode. We define $[N] := \{1, \ldots, N\}$, for all $N \in \mathbb{N}$, and throughout the paper use $t \in [H]$ and $k \in [K]$ to denote time-step inside an episode and the index of an episode, respectively.

A deterministic policy $\pi : \mathcal{S} \times [H] \to \mathcal{A}$ is a mapping from states and time-step indices to actions. We denote by $a_t := \pi(s_t, t)$, the action taken at time t at state s_t according to a policy π. The quality

[2]Similarly to previous work in the finite horizon setting, we state the regret in terms of the horizon H. The regret in the infinite horizon setting is $DS\sqrt{AT}$, where D is the diameter of the MDP.

of a policy π from state s at time t is measured by its value function, which is defined as

$$V_t^\pi(s) := \mathbb{E}\left[\sum_{t'=t}^{H} r(s_{t'}, \pi(s_{t'}, t')) \mid s_t = s\right],$$

where the expectation is over the environment's randomness. An optimal policy maximizes this value for all states s and time-steps t, and the corresponding optimal value is denoted by $V_t^*(s) := \max_\pi V_t^\pi(s)$, for all $t \in [H]$. The optimal value satisfies the optimal Bellman equation, i.e.,

$$V_t^*(s) = T^* V_{t+1}^*(s) := \max_a \left\{ r(s,a) + p(\cdot \mid s,a)^T V_{t+1}^* \right\}. \tag{1}$$

We consider an agent that repeatedly interacts with an MDP in a sequence of episodes $[K]$. The performance of the agent is measured by its *regret*, defined as $\mathrm{Regret}(K) := \sum_{k=1}^{K} \left(V_1^*(s_1^k) - V_1^{\pi_k}(s_1^k) \right)$. Throughout this work, the policy π_k is computed by a 1-step planning operation with respect to the value function estimated by the algorithm at the end of episode $k-1$, denoted by $\bar{V}^{k-1}$. We also call such policy a *greedy policy*. Moreover, s_t^k and a_t^k stand, respectively, for the state and the action taken at the t^{th} time-step of the k^{th} episode.

Next, we define the filtration $\mathcal{F}_k$ that includes all events (states, actions, and rewards) until the end of the k^{th} episode, as well as the initial state of the episode $k+1$. We denote by $T = KH$, the total number of time-steps (samples). Moreover, we denote by $n_k(s,a)$, the number of times that the agent has visited state-action pair (s,a), and by $\hat{X}_k$, the empirical average of a random variable X. Both quantities are based on experience gathered until the end of the k^{th} episode and are $\mathcal{F}_k$ measurable. We also define the probability to visit the state-action pair (s,a) at the k^{th} episode at time-step t by $w_{tk}(s,a) = \Pr\left(s_t^k = s, a_t^k = a \mid s_0^k, \pi_k\right)$. We note that π_k is $\mathcal{F}_{k-1}$ measurable, and thus, $w_{tk}(s,a) = \Pr\left(s_t^k = s, a_t^k = a \mid \mathcal{F}_{k-1}\right)$. Also denote $w_k(s,a) = \sum_{t=1}^{H} w_{tk}(s,a)$.

We use $\tilde{\mathcal{O}}(X)$ to refer to a quantity that depends on X up to poly-log expression of a quantity at most polynomial in S, A, T, K, H, and $\frac{1}{\delta}$. Similarly, $\lesssim$ represents $\leq$ up to numerical constants or poly-log factors. We define $\|X\|_{2,p} := \sqrt{\mathbb{E}_p X^2}$, where p is a probability distribution over the domain of X, and use $X \vee Y := \max\{X, Y\}$. Lastly, $\mathcal{P}(\mathcal{S})$ is the set of probability distributions over the state space $\mathcal{S}$.

3 Real-Time Dynamic Programming

Algorithm 1 Real-Time Dynamic Programming

Initialize: $\forall s \in \mathcal{S},\ \forall t \in [H],\ \bar{V}_t^0(s) = H - (t-1)$.
for $k = 1, 2, \ldots$ **do**
  Initialize s_1^k
  for $t = 1, \ldots, H$ **do**
    $a_t^k \in \arg\max_a r(s_t^k, a) + p(\cdot \mid s_t^k, a)^T \bar{V}_{t+1}^{k-1}$
    $\bar{V}_t^k(s_t^k) = r(s_t^k, a_t^k) + p(\cdot \mid s_t^k, a_t^k)^T \bar{V}_{t+1}^{k-1}$
    Act with a_t^k and observe s_{t+1}^k.
  end for
end for

RTDP [Barto et al., 1995] is a well-known algorithm that solves an MDP when a model of the environment is given. Unlike, e.g., Value Iteration (VI) [Bertsekas and Tsitsiklis, 1996] that solves an MDP by offline calculations, RTDP solves an MDP in a real-time manner. As mentioned in Barto et al. [1995], RTDP can be interpreted as an asynchronous VI adjusted to a real-time algorithm.

Algorithm 1 contains the pseudocode of RTDP for finite-horizon MDPs. The value function is initialized with an optimistic value, i.e., an upper bound of the optimal value. At each time-step t and episode k, the agent acts from the current state s_t^k greedily with respect to the current value at the next time step, $\bar{V}_{t+1}^{k-1}$. It then updates the value of s_t^k according to the optimal Bellman operator. We denote by $\bar{V}$, the value function, and as we show in the following, it always upper bounds V^*. Note that since the action at a fixed state is chosen according to $\bar{V}^{k-1}$, then π_k is $\mathcal{F}_{k-1}$ measurable.

Since RTDP is an online algorithm, i.e., it updates its value estimates through interactions with the environment, it is natural to measure its performance in terms of the regret. The rest of this section is devoted to supplying expected and high-probability bounds on the regret of RTDP, which will also lead to PAC bounds for this algorithm. In Section 4, based on the observations from this section, we will establish minimax regret bounds for 1-step greedy model-based RL.

We start by stating two basic properties of RTDP in the following lemma: the value is always optimistic and decreases in k (see proof in Appendix B). Although the first property is known [Barto et al., 1995], to the best of our knowledge, the second one has not been proven in previous work.

Lemma 1. *For all s, t, and k, it holds that (i) $V_t^*(s) \leq \bar{V}_t^k(s)$ and (ii) $\bar{V}_t^k(s) \leq \bar{V}_t^{k-1}(s)$.*

The following lemma, that we believe is new, relates the difference between the optimistic value $\bar{V}_1^{k-1}(s_1^k)$ and the real value $V_1^{\pi_k}(s_1^k)$ to the *expected cumulative update* of the value function at the end of the k^{th} episode (see proof in Appendix B).

Lemma 2 (Value Update for Exact Model)**.** *The expected cumulative value update of RTDP at the k^{th} episode satisfies*

$$\bar{V}_1^{k-1}(s_1^k) - V_1^{\pi_k}(s_1^k) = \sum_{t=1}^{H} \mathbb{E}[\bar{V}_t^{k-1}(s_t^k) - \bar{V}_t^k(s_t^k) \mid \mathcal{F}_{k-1}].$$

The result relates the difference of the optimistic value $\bar{V}^{k-1}$ and the value of the greedy policy V^{π_k} to the expected update along the trajectory, created by following π_k. Thus, for example, if the optimistic value is overestimated, then the value update throughout this episode is expected to be large.

3.1 Regret and PAC Analysis

Using Lemma 1, we observe that the sequence of values is decreasing and bounded from below. Thus, intuitively, the decrements of the values cannot be indefinitely large. Importantly, Lemma 2 states that when the expected decrements of the values are small, then $V_1^{\pi_k}(s_1^k)$ is close to $\bar{V}^{k-1}(s_1^k)$, and thus, to V^*, since $\bar{V}^{k-1}(s_1^k) \geq \bar{V}^*(s_1^k) \geq V_1^{\pi_k}(s_1^k)$.

Building on this reasoning, we are led to establish a general result on a decreasing process. This result will allow us to formally justify the aforementioned reasoning and derive regret bounds for RTDP. The proof utilizes self-normalized concentration bounds [de la Peña et al., 2007], applied on martingales, and can be found in Appendix A.

Definition 1 (Decreasing Bounded Process)**.** *We call a random process $\{X_k, \mathcal{F}_k\}_{k\geq 0}$, where $\{\mathcal{F}_k\}_{k\geq 0}$ is a filtration and $\{X_k\}_{k\geq 0}$ is adapted to this filtration, a Decreasing Bounded Process, if it satisfies the following properties:*

1. *$\{X_k\}_{k\geq 0}$ decreases, i.e., $X_{k+1} \leq X_k$ a.s. .*
2. *$X_0 = C \geq 0$, and for all k, $X_k \geq 0$ a.s. .*

Theorem 3 (Regret Bound of a Decreasing Bounded Process)**.** *Let $\{X_k, \mathcal{F}_k\}_{k\geq 0}$ be a Decreasing Bounded Process and $R_K = \sum_{k=1}^{K} X_{k-1} - \mathbb{E}[X_k \mid \mathcal{F}_{k-1}]$ be its K-round regret. Then,*

$$\Pr\left\{\exists K > 0 : R_K \geq C\Big(1 + 2\sqrt{\ln(2/\delta)}\Big)^2\right\} \leq \delta.$$

Specifically, it holds that $\Pr\{\exists K > 0 : R_K \geq 9C\ln(3/\delta)\} \leq \delta$.

We are now ready to prove the central result of this section, the expected and high-probability regret bounds on RTDP (see full proof in Appendix B).

Theorem 4 (Regret Bounds for RTDP)**.** *The following regret bounds hold for RTDP:*

1. *$\mathbb{E}[\mathrm{Regret}(K)] \leq SH^2$.*
2. *For any $\delta > 0$, with probability $1-\delta$, for all $K > 0$, $\mathrm{Regret}(K) \leq 9SH^2\ln(3SH/\delta)$.*

Proof Sketch. We give a sketch of the proof of the second claim. Applying Lemmas 1 and then 2,

$$\begin{aligned}\text{Regret}(K) &:= \sum_{k=1}^{K} V_1^*(s_1^k) - V_1^{\pi_k}(s_1^k) \leq \sum_{k=1}^{K} \bar{V}_1^{k-1}(s_1^k) - V^{\pi_k}(s_1^k) \\ &\leq \sum_{k=1}^{K}\sum_{t=1}^{H} \mathbb{E}[\bar{V}_t^{k-1}(s_t^k) - \bar{V}_t^k(s_t^k) \mid \mathcal{F}_{k-1}]. \end{aligned} \tag{2}$$

We then establish (see Lemma 34) that RHS of (2) is, in fact, a sum of SH Decreasing Bounded Processes, i.e.,

$$(2) = \sum_{t=1}^{H}\sum_{s\in\mathcal{S}}\sum_{k=1}^{K} \bar{V}_t^{k-1}(s) - \mathbb{E}[\bar{V}_t^k(s) \mid \mathcal{F}_{k-1}]. \tag{3}$$

Since for any fixed s, t, $\left\{\bar{V}_t^k(s)\right\}_{k\geq 0}$ is a decreasing process by Lemma 1, we can use Theorem 3, for a fixed s, t, and conclude the proof by applying the union bound on all SH terms in (3). □

Theorem 4 exhibits a regret bound that does not depend on $T = KH$. While it is expected that RTDP, that has access to the exact model, would achieve better performance than an RL algorithm with no such access, a regret bound independent of T is a noteworthy result. Indeed, it leads to the following Uniform PAC (see Dann et al. 2017 for the definition) and $(0, \delta)$ PAC guarantees for RTDP (see proofs in Appendix B). To the best of our knowledge, both are the first PAC guarantees for RTDP.[3]

Corollary 5 (RTDP is Uniform PAC)**.** *Let $\delta > 0$ and N_ϵ be the number of episodes in which RTDP outputs a policy with $V_1^*(s_1^k) - V_1^{\pi_k}(s_1^k) > \epsilon$. Then,*

$$\Pr\left\{\exists \epsilon > 0 : N_\epsilon \geq \frac{9SH^2\ln(3SH/\delta)}{\epsilon}\right\} \leq \delta.$$

Corollary 6 (RTDP is $(0, \delta)$ PAC)**.** *Let $\delta > 0$ and N be the number of episodes in which RTDP outputs a non optimal policy. Define the (unknown) gap of the MDP, $\Delta(\mathcal{M}) = \min_s \min_{\pi: V_1^\pi(s) \neq V_1^*(s)} V_1^*(s) - V_1^\pi(s) > 0$. Then,*

$$\Pr\left\{N \geq \frac{9SH^2\ln(3SH/\delta)}{\Delta(\mathcal{M})}\right\} \leq \delta.$$

4 Exploration in Model-based RL: Greedy Policy Achieves Minimax Regret

We start this section by formulating a general optimistic RL scheme that acts by 1-step planning (see Algorithm 2). Then, we establish Lemma 7, which generalizes Lemma 2 to the case where a non-exact model is used for the value updates. Using this lemma, we offer a novel regret decomposition for algorithms which follow Algorithm 2. Based on the decomposition, we analyze generalizations of UCRL2 [Jaksch et al., 2010] (for finite horizon MDPs) and EULER [Zanette and Brunskill, 2019], that use greedy policies instead of solving an MDP (full planning) at the beginning of each episode. Surprisingly, we find that both generalized algorithms do not suffer from performance degradation, up to numerical constants and logarithmic factors. Thus, we conclude that there exists an RL algorithm that achieves the minimax regret bound, while acting according to greedy policies.

Consider the general RL scheme that explores by greedy policies as depicted in Algorithm 2. The value $\bar{V}$ is initialized optimistically and the algorithm interacts with the unknown environment in an episodic manner. At each time-step t, a greedy policy from the current state, s_t^k, is calculated optimistically based on the empirical model $(\hat{r}_{k-1}, \hat{p}_{k-1}, n_{k-1})$ and the current value at the next time-step $\bar{V}_{t+1}^{k-1}$. This is done in a subroutine called 'ModelBasedOptimisticQ'.[4] We further assume the optimistic Q-function has the form $\bar{Q}(s_t^k, a) = \tilde{r}_{k-1}(s_t^k, a) + \tilde{p}_{k-1}(\cdot \mid s_t^k, a)^T \bar{V}_{t+1}^{k-1}$ and refer to

[3] Existing PAC results on RTDP analyze variations of RTDP in which ϵ is an input parameter of the algorithm.

[4] We also allow the subroutine to use $\mathcal{O}(S)$ internal memory for auxiliary calculations, which does not change the overall space complexity.

Algorithm 2 Model-based RL with Greedy Policies

1: Initialize: $\forall s \in \mathcal{S},\ \forall t \in [H],\ \bar{V}_t^0(s) = H - (t-1)$.
2: **for** $k = 1, 2, \ldots$ **do**
3: Initialize s_1^k
4: **for** $t = 1, \ldots, H$ **do**
5: $\forall a,\ \bar{Q}(s_t^k, a) = \text{ModelBasedOptimisticQ}\big(\hat{r}_{k-1}, \hat{p}_{k-1}, n_{k-1}, \bar{V}_{t+1}^{k-1}\big)$
6: $a_t^k \in \arg\max_a \bar{Q}(s_t^k, a)$
7: $\bar{V}_t^k(s_t^k) = \min\big\{\bar{V}_t^{k-1}(s_t^k), \bar{Q}(s_t^k, a_t^k)\big\}$
8: Act with a_t^k and observe s_{t+1}^k.
9: **end for**
10: Update $\hat{r}_k, \hat{p}_k, n_k$ with all experience gathered in episode.
11: **end for**

$(\tilde{r}_{k-1}, \tilde{p}_{k-1})$ as the optimistic model. The agent interacts with the environment based on the greedy policy with respect to $\bar{Q}$ and uses the gathered experience to update the empirical model at the end of the episode.

By construction of the update rule (see Line 7), the value is a decreasing function of k, for all $(s,t) \in \mathcal{S} \times [H]$. Thus, property (ii) in Lemma 1 holds for Algorithm 2. Furthermore, the algorithms analyzed in this section will also be optimistic with high probability, i.e., property (i) in Lemma 1 also holds. Finally, since the value update uses the empirical quantities $\hat{r}_{k-1}$, $\hat{p}_{k-1}$, n_{k-1} and $\bar{V}_{t+1}^{k-1}$ from the previous episode, policy π_k is still $\mathcal{F}_{k-1}$ measurable.

The following lemma generalizes Lemma 2 to the case where, unlike in RTDP, the update rule does not use the exact model (see proof in Appendix C).

Lemma 7 (Value Update for Optimistic Model)**.** *The expected cumulative value update of Algorithm 2 in the k^{th} episode is bounded by*

$$\bar{V}_1^{k-1}(s_1^k) - V_1^{\pi_k}(s_1^k) \le \sum_{t=1}^{H} \mathbb{E}\big[\bar{V}_t^{k-1}(s_t^k) - \bar{V}_t^k(s_t^k) \mid \mathcal{F}_{k-1}\big] + \sum_{t=1}^{H} \mathbb{E}\big[(\tilde{r}_{k-1} - r)(s_t^k, a_t^k) + (\tilde{p}_{k-1} - p)(\cdot \mid s_t^k, a_t^k)^T \bar{V}_{t+1}^{k-1} \mid \mathcal{F}_{k-1}\big] \ .$$

In the rest of the section, we consider two instantiations of the subroutine 'ModelBasedOptimisticQ' in Algorithm 2. We use the bonus terms of UCRL2 and of EULER to acquire an optimistic Q-function, $\bar{Q}$. These two options then lead to UCRL2 with Greedy Policies (UCRL2-GP) and EULER with Greedy Policies (EULER-GP) algorithms.

4.1 UCRL2 with Greedy Policies for Finite-Horizon MDPs

Algorithm 3 UCRL2 with Greedy Policies (UCRL2-GP)

1: $\tilde{r}_{k-1}(s_t^k, a) = \hat{r}_{k-1}(s_t^k, a) + \sqrt{\frac{2 \ln \frac{8SAT}{\delta}}{n_{k-1}(s_t^k, a) \vee 1}}$
2: $CI(s_t^k, a) = \left\{ P' \in \mathcal{P}(\mathcal{S}) : \|P'(\cdot) - \hat{p}_{k-1}(\cdot \mid s_t^k, a)\|_1 \le \sqrt{\frac{4S \ln \frac{12SAT}{\delta}}{n_{k-1}(s_t^k, a) \vee 1}} \right\}$
3: $\tilde{p}_{k-1}(\cdot \mid s_t^k, a) = \arg\max_{P' \in CI(s_t^k, a)} P'(\cdot \mid s_t^k, a)^T \bar{V}_{t+1}^{k-1}$
4: $\bar{Q}(s_t^k, a) = \tilde{r}_{k-1}(s_t^k, a) + \tilde{p}_{k-1}(\cdot \mid s_t^k, a)^T \bar{V}_{t+1}^{k-1}$
5: **Return** $\bar{Q}(s_t^k, a)$

We form the optimistic local model based on the confidence set of UCRL2 [Jaksch et al., 2010]. This amounts to use Algorithm 3 as the subroutine 'ModelBasedOptimisticQ' in Algorithm 2. The maximization problem on Line 3 of Algorithm 3 is common, when using bonus based on an optimistic model [Jaksch et al., 2010], and it can be solved efficiently in $\tilde{\mathcal{O}}(\mathcal{N})$ operations (e.g., Strehl and Littman 2008, Section 3.1.5). A full version of the algorithm can be found in Appendix D.

Thus, Algorithm 3 performs $\mathcal{N}AH$ operations per episode. This saves the need to perform Extended Value Iteration [Jaksch et al., 2010], that costs $\mathcal{N}SAH$ operations per episode (an extra factor of S). Despite the significant improvement in terms of computational complexity, the regret of UCRL2-GP is similar to the one of UCRL2 [Jaksch et al., 2010] as the following theorem formalizes (see proof in Appendix D).

Theorem 8 (Regret Bound of UCRL2-GP)**.** *For any time $T \leq KH$, with probability at least $1-\delta$, the regret of UCRL2-GP is bounded by $\tilde{\mathcal{O}}\Big(HS\sqrt{AT} + H^2\sqrt{S}SA\Big)$.*

Proof Sketch. Using the optimism of the value function (see Section D.2) and by applying Lemma 7, we bound the regret as follows:

$$\begin{aligned}\text{Regret}(K) &= \sum_{k=1}^{K} V_1^*(s_1^k) - V_1^{\pi_k}(s_1^k) \leq \sum_{k=1}^{K} \bar{V}_1^{k-1}(s_1^k) - V_1^{\pi_k}(s_1^k) \\ &\leq \sum_{k=1}^{K}\sum_{t=1}^{H} \mathbb{E}[\bar{V}_t^{k-1}(s_t^k) - \bar{V}_t^{k}(s_t^k) \mid \mathcal{F}_{k-1}] \\ &\quad + \sum_{k=1}^{K}\sum_{t=1}^{H} \mathbb{E}\left[(\tilde{r}_{k-1} - r)(s_t^k, a_t^k) + (\tilde{p}_{k-1} - p)(\cdot \mid s_t^k, a_t^k)^T \bar{V}_{t+1}^{k-1} \mid \mathcal{F}_{k-1}\right]. \qquad (4)\end{aligned}$$

Thus, the regret is upper bounded by two terms. As in Theorem 4, by applying Lemma 11 (Appendix A), the first term in (4) is a sum of SH Decreasing Bounded Processes, and can thus be bounded by $\tilde{\mathcal{O}}\big(SH^2\big)$. The presence of the second term in (4) is common in recent regret analyses (e.g., Dann et al. 2017). Using standard techniques [Jaksch et al., 2010, Dann et al., 2017, Zanette and Brunskill, 2019], this term can be bounded (up to additive constant factors) with high probability by $\lesssim H\sqrt{S}\sum_{k=1}^{K}\sum_{t=1}^{H}\mathbb{E}\left[\sqrt{\frac{1}{n_{k-1}(s_t^k,a_t^k)}} \mid \mathcal{F}_{k-1}\right] \leq \tilde{\mathcal{O}}(HS\sqrt{AT})$. □

4.2 EULER with Greedy Policies

In this section, we use bonus terms as in EULER [Zanette and Brunskill, 2019]. Similar to the previous section, this amounts to replacing the subroutine 'ModelBasedOptimisticQ' in Algorithm 2 with a subroutine based on the bonus terms from [Zanette and Brunskill, 2019]. Algorithm 5 in Appendix E contains the pseudocode of the algorithm. The bonus terms in EULER are based on the empirical Bernstein inequality and tracking both an upper bound $\bar{V}_t$ and a lower-bound $\underline{V}_t$ on V_t^*. Using these, EULER achieves both minimax optimal and problem dependent regret bounds.

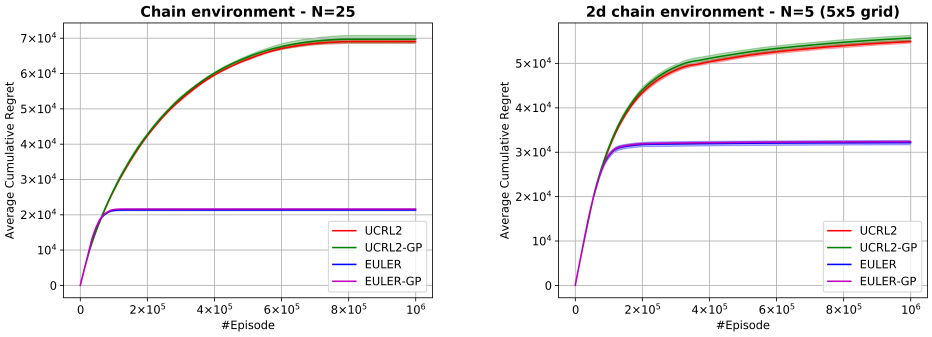
EULER [Zanette and Brunskill, 2019] performs $\mathcal{O}(\mathcal{N}SAH)$ computations per episode (same as the VI algorithm), while EULER-GP requires only $\mathcal{O}(\mathcal{N}AH)$. Despite this advantage in computational complexity, EULER-GP exhibits similar minimax regret bounds to EULER (see proof in Appendix E), much like the equivalent performance of UCRL2 and UCRL2-GP proved in Section 4.1.

Theorem 9 (Regret Bound of EULER-GP)**.** *Let $\mathcal{G}$ be an upper bound on the total reward collected within an episode. Define $\mathbb{Q}^* := \max_{s,a,t}\big(\text{Var}R(s,a) + \text{Var}_{s'\sim p(\cdot|s,a)}V_{t+1}^*(s)\big)$ and $H_{\text{eff}} := \min\big\{\mathbb{Q}^*, \mathcal{G}^2/H\big\}$. With probability $1-\delta$, for any time $T \leq KH$ jointly on all episodes $k \in [K]$, the regret of EULER-GP is bounded by $\tilde{\mathcal{O}}\Big(\sqrt{H_{\text{eff}}SAT} + \sqrt{S}SAH^2(\sqrt{S}+\sqrt{H})\Big)$. Thus, it is also bounded by $\tilde{\mathcal{O}}\Big(\sqrt{HSAT} + \sqrt{S}SAH^2(\sqrt{S}+\sqrt{H})\Big)$.*

Note that Theorem 9 exhibits similar problem-dependent regret-bounds as in Theorem 1 of [Zanette and Brunskill, 2019]. Thus, the same corollaries derived in [Zanette and Brunskill, 2019] for EULER can also be applied to EULER-GP.

5 Experiments

In this section, we present an empirical evaluation of both UCRL2 and EULER, and compare their performance to the proposed variants, which use greedy policy updates, UCRL2-GP and EULER-GP,



(a) Chain environment with $N = 25$ states (b) 2D chain environment with 5×5 grid

Figure 1: A comparison UCRL2 and EULER with their greedy counterpart. Results are averaged over 5 random seeds and are shown alongside error bars ($\pm$3std).

respectively. We evaluated the algorithms on two environments. (i) **Chain environment** [Osband and Van Roy, 2017]: In this MDP, there are N states, which are connected in a chain. The agent starts at the left side of the chain and can move either to the left or try moving to the right, which succeeds w.p. $1 - 1/N$, and results with movement to the left otherwise. The agent goal is to reach the right side of the chain and try moving to the right, which results with a reward $r \sim \mathcal{N}(1, 1)$. Moving backwards from the initials state also results with $r \sim \mathcal{N}(0, 1)$, and otherwise, the reward is $r = 0$. Furthermore, the horizon is set to $H = N$, so that the agent must always move to the right to have a chance to receive a reward. (ii) **2D chain**: A generalization of the chain environment, in which the agent starts at the upper-left corner of a $N \times N$ grid and aims to reach the lower-right corner and move towards this corner, in $H = 2N - 1$ steps. Similarly to the chain environment, there is a probability $1/H$ to move backwards (up or left), and the agent must always move toward the corner to observe a reward $r \sim \mathcal{N}(1, 1)$. Moving into the starting corner results with $r \sim \mathcal{N}(0, 1)$, and otherwise $r = 0$. This environment is more challenging for greedy updates, since there are many possible trajectories that lead to reward.

The simulation results can be found in Figure 1, and clearly indicate that using greedy planning leads to negligible degradation in the performance. Thus, the simulations verify our claim that greedy policy updates greatly improve the efficiency of the algorithm while maintaining the same performance.

6 Related Work

Real-Time Dynamic Programming: RTDP [Barto et al., 1995] has been extensively used and has many variants that exhibit superior empirical performance (e.g., [Bonet and Geffner, 2003, McMahan et al., 2005, Smith and Simmons, 2006]). For discounted MDPs, Strehl et al. [2006] proved (ϵ, δ)-PAC bounds of $\tilde{O}\left(SA/\epsilon^2(1-\gamma)^4\right)$, for a modified version of RTDP in which the value updates occur only if the decrease in value is larger than $\epsilon(1-\gamma)$. I.e., their algorithm explicitly use ϵ to mark states with accurate value estimate. We prove that RTDP converges in a rate of $\tilde{O}\left(SH^2/\epsilon\right)$ without knowing ϵ. Indeed, Strehl et al. [2006] posed *whether the original RTDP is PAC* as an open problem. Furthermore, no regret bound for RTDP has been reported in the literature.

Regret bounds for RL: The most renowned algorithms with regret guarantees for undiscounted infinite-horizon MDPs are UCRL2 [Jaksch et al., 2010] and REGAL [Bartlett and Tewari, 2009], which have been extended throughout the years (e.g., by Fruit et al. 2018, Talebi and Maillard 2018). Recently, there is an increasing interest in regret bounds for MDPs with finite horizon H and stationary dynamics. In this scenario, UCRL2 enjoys a regret bound of order $HS\sqrt{AT}$. Azar et al. [2017] proposed UCBVI, with improved regret bound of order $\sqrt{HSAT}$, which is also asymptotically tight [Osband and Van Roy, 2016]. Dann et al. [2018] presented ORLC that achieves tight regret bounds and (nearly) tight PAC guarantees for non-stationary MDPs. Finally, Zanette and Brunskill [2019] proposed EULER, an algorithm that enjoys tight minimax regret bounds and has additional

problem-dependent bounds that encapsulate the MDP's complexity. All of these algorithms are model-based and require full-planning. Model-free RL was analyzed by [Jin et al., 2018]. There, the authors exhibit regret bounds that are worse by a factor of H relatively to the lower-bound. To the best of our knowledge, there are no model-based algorithms with regret guarantees that avoid full-planning. It is worth noting that while all the above algorithms, and the ones in this work, rely on the Optimism in the Face of Uncertainty principle [Lai and Robbins, 1985], Thompson Sampling model-based RL algorithms exist [Osband et al., 2013, Gopalan and Mannor, 2015, Agrawal and Jia, 2017, Osband and Van Roy, 2017]. There, a model is sampled from a distribution over models, on which full-planning takes place.

Greedy policies in model-based RL: By adjusting RTDP to the case where the model is unknown, Strehl et al. [2012] formulated model-based RL algorithms that act using a greedy policy. They proved a $\tilde{O}\left(S^2A/\epsilon^3(1-\gamma)^6\right)$ sample complexity bound for discounted MDPs. To the best of our knowledge, there are no regret bounds for model-based RL algorithms that act by greedy policies.

Practical model-based RL: Due to the high computational complexity of planning in model-based RL, most of the practical algorithms are model-free (e.g., Mnih et al. 2015). Algorithms that do use a model usually only take advantage of local information. For example, Dyna [Sutton, 1991, Peng et al., 2018] selects state-action pairs, either randomly or via prioritized sweeping [Moore and Atkeson, 1993, Van Seijen and Sutton, 2013], and updates them according to a local model. Other papers use the local model to plan for a short horizon from the current state [Tamar et al., 2016, Hafner et al., 2018]. The performance of such algorithms depends heavily on the planning horizon, that in turn dramatically increases the computational complexity.

7 Conclusions and Future Work

In this work, we established that tabular model-based RL algorithms can explore by 1-step planning instead of full-planning, without suffering from performance degradation. Specifically, exploring with model-based greedy policies can be minimax optimal in terms of regret. Differently put, the variance caused by exploring with greedy policies is smaller than the variance caused by learning a sufficiently good model. Indeed, the extra term which appears due to the greedy exploration is $\tilde{O}(SH^2)$ (e.g., the first term in (4)); a constant term, smaller than the existing constant terms of UCRL2 and EULER.

This work raises and highlights some interesting research questions. The obvious ones are extensions to average and discounted MDPs, as well as to Thompson sampling based RL algorithms. Although these scenarios are harder or different in terms of analysis, we believe this work introduces the relevant approach to tackle this question. Another interesting question is the applicability of the results in large-scale problems, when tabular representation is infeasible and approximation must be used. There, algorithms that act using lookahead policies, instead of 1-step planning, are expected to yield better performance, as they are less sensitive to value approximation errors (e.g., Bertsekas and Tsitsiklis 1996, Jiang et al. 2018, Efroni et al. 2018b,a). Even then, full-planning, as opposed to using a short-horizon planning, might be unnecessary. Lastly, establishing whether the model-based approach is or is not provably better than the model-free approach, as the current state of the literature suggests, is yet an important and unsolved open problem.

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A Proofs on Decreasing Bounded Processes

In this section, we state and prove useful results on Decreasing Bounded Processes (see Definition 1). These results will be in use in proofs of the central theorems of this work.

Theorem 3 (Regret Bound of a Decreasing Bounded Process)**.** *Let $\{X_k, \mathcal{F}_k\}_{k\geq 0}$ be a Decreasing Bounded Process and $R_K = \sum_{k=1}^{K} X_{k-1} - \mathbb{E}[X_k \mid \mathcal{F}_{k-1}]$ be its K-round regret. Then,*

$$\Pr\left\{\exists K > 0 : R_K \geq C\left(1 + 2\sqrt{\ln(2/\delta)}\right)^2\right\} \leq \delta.$$

Specifically, it holds that $\Pr\{\exists K > 0 : R_K \geq 9C\ln(3/\delta)\} \leq \delta$.

Proof. Without loss of generality, assume $C > 0$, since otherwise the results are trivial. We start by remarking that R_K is almost surely monotonically increasing, since $X_k \leq X_{k-1}$. Define the martingale difference process

$$\xi_k = X_k - \mathbb{E}[X_k \mid \mathcal{F}_{k-1}] = X_k - X_{k-1} - \mathbb{E}[X_k - X_{k-1} \mid \mathcal{F}_{k-1}]$$

and the martingale process $M_K = \sum_{k=1}^{K} \xi_k$. Since $X_k \geq 0$ almost surely, R_K can be bounded by $R_K = M_K + X_0 - X_K \leq X_0 + M_K$. Also define the quadratic variations as $\langle M\rangle_K = \sum_{k=1}^{K} \mathbb{E}\left[\xi_k^2 \mid \mathcal{F}_{k-1}\right]$ and $[M]_K = \sum_{k=1}^{K} \xi_k^2$. Next, recall Theorem 2.7 of [de la Peña et al., 2007]:

Theorem 10. *Let A and B be two random variables, such that for all $\lambda \in \mathbb{R}$, we have*

$$\mathbb{E}\left[e^{\lambda A - \frac{\lambda^2 B^2}{2}}\right] \leq 1 \ . \tag{5}$$

Then, $\forall x > 0$,

$$\Pr\left\{\frac{|A|}{\sqrt{B^2 + \mathbb{E}[B^2]}} > x\right\} \leq \sqrt{2}e^{-x^2/4}. \tag{6}$$

Condition (5) holds for $A_K = M_K$ and $B_K^2 = \langle M\rangle_K + [M]_K$, due to Theorem 9.21 of [de la Peña et al., 2008]. A_K can be easily bounded by $|A_K| \geq R_K - X_0 \geq R_K - C$. To bound B_K^2, we first calculate ξ_k^2 and $\mathbb{E}\left[\xi_k^2 \mid \mathcal{F}_{k-1}\right]$:

$$\xi_k^2 = (X_k - X_{k-1})^2 - 2(X_k - X_{k-1})\mathbb{E}[X_k - X_{k-1} \mid \mathcal{F}_{k-1}] + (\mathbb{E}[X_k - X_{k-1} \mid \mathcal{F}_{k-1}])^2,$$
$$\mathbb{E}\left[\xi_k^2 \mid \mathcal{F}_{k-1}\right] = \mathbb{E}\left[(X_k - X_{k-1})^2 \mid \mathcal{F}_{k-1}\right] - (\mathbb{E}[X_k - X_{k-1} \mid \mathcal{F}_{k-1}])^2.$$

Thus,

$$\begin{aligned}
&\xi_k^2 + \mathbb{E}\left[\xi_k^2 \mid \mathcal{F}_{k-1}\right] \\
&= (X_k - X_{k-1})^2 + \mathbb{E}\left[(X_k - X_{k-1})^2 \mid \mathcal{F}_{k-1}\right] - 2(X_k - X_{k-1})\mathbb{E}[X_k - X_{k-1} \mid \mathcal{F}_{k-1}] \\
&\overset{(*)}{\leq} (X_k - X_{k-1})^2 + \mathbb{E}\left[(X_k - X_{k-1})^2 \mid \mathcal{F}_{k-1}\right] \\
&\overset{(**)}{\leq} (X_k - X_{k-1})^2 + C\mathbb{E}[X_{k-1} - X_k \mid \mathcal{F}_{k-1}] \ .
\end{aligned}$$

In $(*)$ we used the fact that $X_{k-1} - X_k \geq 0$ a.s., which allows us to conclude that the cross-term is non-positive. In (**), we bounded $0 \leq X_{k-1} - X_k \leq C$. We can also bound $\sum_{k=1}^{K}(X_{k-1} - X_k)^2 \leq C^2$, since each of the summands is a.s. non-negative, and thus,

$$\sum_{k=1}^{K}(X_{k-1} - X_k)^2 \leq \left(\sum_{k=1}^{K} X_{k-1} - X_k\right)^2 = (X_K - X_0)^2 \leq C^2.$$

Combining all of the above bounds yields

$$\begin{aligned}
B_K^2 &\leq \sum_{k=1}^{K}\left((X_k - X_{k-1})^2 + C\mathbb{E}[X_{k-1} - X_k \mid \mathcal{F}_{k-1}]\right) \\
&\leq C^2 + C\sum_{k=1}^{K}\mathbb{E}[X_{k-1} - X_k \mid \mathcal{F}_{k-1}] = C^2 + CR_K.
\end{aligned}$$

Finally, we can bound $\mathbb{E}\big[B_K^2\big]$ by

$$\begin{aligned}\mathbb{E}\big[B_K^2\big] &= \sum_{k=1}^{K}\mathbb{E}\big[\xi_k^2 + \mathbb{E}\big[\xi_k^2 \mid \mathcal{F}_{k-1}\big]\big] = 2\sum_{k=1}^{K}\mathbb{E}\big[\mathbb{E}\big[\xi_k^2 \mid \mathcal{F}_{k-1}\big]\big] \\ &= 2\sum_{k=1}^{K}\mathbb{E}\Big[\mathbb{E}\Big[(X_k - X_{k-1})^2 \mid \mathcal{F}_{k-1}\Big] - \left(\mathbb{E}[X_k - X_{k-1} \mid \mathcal{F}_{k-1}]\right)^2\Big] \\ &\le 2\sum_{k=1}^{K}\mathbb{E}\Big[\mathbb{E}\Big[(X_k - X_{k-1})^2 \mid \mathcal{F}_{k-1}\Big]\Big] = 2\sum_{k=1}^{K}(X_k - X_{k-1})^2 \le 2C^2.\end{aligned}$$

Combining everything we obtain

$$\frac{|A|}{\sqrt{B^2 + \mathbb{E}[B^2]}} \ge \frac{R_K - C}{\sqrt{C^2 + CR_K + 2C^2}} = \frac{R_K - C}{\sqrt{3C^2 + CR_K}}.$$

Or, substituting in (6), we have

$$\Pr\left\{\frac{R_K - C}{\sqrt{3C^2 + CR_K}} > x\right\} \le \Pr\left\{\frac{|A|}{\sqrt{B^2 + \mathbb{E}[B^2]}} > x\right\} \le \sqrt{2}e^{-x^2/4}.$$

Next, notice that for $C > 0$, the function $f(y) = \frac{y-C}{\sqrt{3C^2+Cy}}$ is monotonically increasing for any $y > 0$:

$$f'(y) = \frac{\sqrt{3C^2 + Cy} - \frac{C(y-C)}{2\sqrt{3C^2+Cy}}}{3C^2 + Cy} = \frac{2(3C^2 + Cy) - Cy + C^2}{2(3C^2 + Cy)^{3/2}} = \frac{7C^2 + Cy}{2(3C^2 + Cy)^{3/2}} > 0$$

Moreover, for $y = C(1+x)^2$,

$$\begin{aligned}f\big(C(1+x)^2\big) &= \frac{C(1+x)^2 - C}{\sqrt{3C^2 + C^2(1+x)^2}} = \frac{Cx^2 + 2Cx}{\sqrt{4C^2 + 2C^2x + C^2x^2}} \\ &> \frac{Cx^2 + 2Cx}{\sqrt{4C^2 + 4C^2x + C^2x^2}} = \frac{Cx^2 + 2Cx}{Cx + 2C} = x \ ,\end{aligned}$$

where the inequality holds since $x > 0$. Thus, if $R_K \ge C(1+x)^2$, then $f(R_K) > x$, and we can bound the probability that $R_K \ge C(1+x)^2$ by

$$\Pr\big\{R_K \ge C(1+x)^2\big\} \le \Pr\left\{\frac{R_K - C}{\sqrt{3C^2 + CR_K}} > x\right\} \le \sqrt{2}e^{-x^2/4} \ ,$$

and setting $x = 2\sqrt{\ln\frac{2}{\delta}} > 0$, we obtain

$$\Pr\left\{R_K \ge C\left(1 + 2\sqrt{\ln\frac{2}{\delta}}\right)^2\right\} \le \delta \ .$$

We remark that since R_K is monotonically increasing a.s., this bound also implies that

$$\Pr\left\{\exists N : 1 \le N \le K,\ R_N \ge C\left(1 + 2\sqrt{\ln\frac{2}{\delta}}\right)^2\right\} \le \delta.$$

To obtain a uniform bound, that is, bound that holds for all $K > 0$, note that the random sequence $Z_K = \mathbb{1}\left\{\exists 1 \leq N \leq K : R_N \geq C\left(1 + 2\sqrt{\ln \frac{2}{\delta}}\right)^2\right\}$ is monotonically increasing in K and bounded. Thus, due to monotone convergence

$$\Pr\left\{\exists K > 0 : R_K \geq C\left(1 + 2\sqrt{\ln \frac{2}{\delta}}\right)^2\right\} = \mathbb{E}\Big[\lim_{K\to\infty} Z_K\Big] = \lim_{K\to\infty} \mathbb{E}[Z_K]$$

$$= \lim_{K\to\infty} \Pr\left\{\exists 1 \leq N \leq K : R_N \geq C\left(1 + 2\sqrt{\ln \frac{2}{\delta}}\right)^2\right\} \leq \delta.$$

To conclude the proof, note that $\delta \leq 1$, and thus, $\ln \frac{3}{\delta} \geq 1$. Therefore, we can bound

$$C\left(1 + 2\sqrt{\ln \frac{2}{\delta}}\right)^2 \leq C\left(1 + 2\sqrt{\ln \frac{3}{\delta}}\right)^2 \leq C\left(3\sqrt{\ln \frac{3}{\delta}}\right)^2 = 9C \ln \frac{3}{\delta} \ ,$$

which yields the second bound. □

Lemma 11. *Let $\{X_n^k\}_{k\geq 1}$ be a Bounded Decreasing Process in $[0, C]$ for any $n \in [N]$. The regret of the sum of processes is defined as $R(K) = \sum_{n=1}^{N} \sum_{k=1}^{K} X_n^{k-1} - \mathbb{E}[X_n^k \mid \mathcal{F}_{k-1}]$. Then, for any $\delta > 0$, we have*

$$\Pr\left\{\exists K > 0 : R(K) \geq 9CN \ln \frac{3N}{\delta}\right\} \leq \delta.$$

Proof. We first remark that if $X_n^0 < C$, we can replace it to $X_n^0 = C$, which only increases the regret, so we assume w.l.o.g. that $X_n^0 = C$. Define

$$R_n(K) := \sum_{k=1}^{K} X_n^{k-1}(s) - \mathbb{E}[X_n^k(s) \mid \mathcal{F}_{k-1}].$$

Define the event $A_n := \left\{\exists K > 0 : R_n(K) \geq 9CN \ln \frac{3N}{\delta}\right\}$. By applying Theorem 3, with probability $\frac{\delta}{N}$, it holds that for a fixed $n \in [N]$

$$\Pr\left\{\exists K > 0 : R_n(K) \geq 9C \ln \frac{3N}{\delta}\right\} = \Pr\{A_n\} \leq \frac{\delta}{N}. \tag{7}$$

Finally, we obtain

$$\Pr\left\{\exists K > 0 : R(K) \geq 9NC \ln \frac{3N}{\delta}\right\} = \Pr\left\{\exists K > 0 : \sum_{n=1}^{N} R_n(K) \geq 9NC \ln \frac{3N}{\delta}\right\}$$

$$\overset{(1)}{\leq} \Pr\left\{\bigcup_{n=1}^{N} A_n\right\} \overset{(2)}{\leq} \sum_{n=1}^{N} \Pr\{A_n\} \overset{(3)}{\leq} \delta.$$

Relation (1) holds since

$$\left\{\exists K > 0 : \sum_{n=1}^{N} R_n(K) \geq 9NC \ln \frac{3N}{\delta}\right\} \subseteq \bigcup_{n=1}^{N} A_n.$$

In (2) we use the union bound and (3) holds by (7). □

B Proof of Real-Time Dynamic Programming Bounds

Lemma 1. *For all s, t, and k, it holds that (i) $V_t^*(s) \leq \bar{V}_t^k(s)$ and (ii) $\bar{V}_t^k(s) \leq \bar{V}_t^{k-1}(s)$.*

Proof. Both claims are proven using induction.

(i) By the initialization, $\forall s, t,\ V_t^*(s) \leq V_t^0(s)$. Assume the claim holds for $k-1$ episodes. Let s_t^k be the state the algorithm is at in the t^{th} time-step of the k^{th} episode. By the value update of Algorithm 1,

$$\begin{aligned}\bar{V}_t^k(s_t^k) &= \max_a\ r(s_t^k, a) + \sum_{s'} p(s' \mid s_t^k, a)\bar{V}_{t+1}^{k-1}(s') \\ &\geq \max_a\ r(s_t^k, a) + \sum_{s'} p(s' \mid s_t^k, a)\bar{V}_{t+1}^{*}(s') = V^*(s_t^k).\end{aligned}$$

The second relation holds by the induction hypothesis and the monotonicity of the optimal Bellman operator [Bertsekas and Tsitsiklis, 1996]. The third relation holds by the recursion satisfied by the optimal value function (see Section 2). Thus, the induction step is proven for the first claim.

(ii) To prove the base case of the second claim we use the optimistic initialization. Let s_t^1 be the state the algorithm is at in the t^{th} time-step of the first episode. By the update rule,

$$\begin{aligned}\bar{V}_t^1(s_t^1) &= \max_a\ r(s_t^1, a) + \sum_{s'} p(s' \mid s_t^1, a)\bar{V}_{t+1}^{0}(s') \\ &\overset{(1)}{=} \max_a\ r(s_t^1, a) + H - t \\ &\overset{(2)}{\leq} 1 + H - t = H - (t-1) \overset{(3)}{=} \bar{V}_t^0(s_t^1).\end{aligned}$$

Relation (1) holds by the initialization of the values, (2) holds since $r(s,a) \in [0,1]$ and (3) is by the initialization. States that were not visited on the first episode were not update, and thus the inequality trivially holds.

Assume the second claim holds for $k-1$ episodes. Let s_t^k be the state that the algorithm is at in the t^{th} time-step of the k^{th} episode. By the value update of Algorithm 1, we have

$$\bar{V}_t^k(s_t^k) = \max_a\ r(s_t^k, a) + \sum_{s'} p(s' \mid s_t^k, a)\bar{V}_{t+1}^{k-1}(s').$$

If s_t^k was previously updated, let $\bar{k}$ be the previous episode in which the update occured. By the induction hypothesis, we have that $\forall s, t,\ \bar{V}_t^{\bar{k}}(s) \geq \bar{V}_t^{k-1}(s)$. Using the monotonicity of the Bellman operator [Bertsekas and Tsitsiklis, 1996], we may write

$$\begin{aligned}&\max_a\ r(s_t^k, a) + \sum_{s'} p(s' \mid s_t^k, a)\bar{V}_{t+1}^{k-1}(s') \\ &\leq \max_a\ r(s_t^k, a) + \sum_{s'} p(s' \mid s_t^k, a)\bar{V}_{t+1}^{\bar{k}-1}(s') = \bar{V}^{k-1}(s_t^k).\end{aligned}$$

Thus, $\bar{V}_t^k(s_t^k) \leq \bar{V}^{k-1}(s_t^k)$ and the induction step is proved. If s_t^k was not previously updated, then $\bar{V}_t^{k-1}(s_t^k) = \bar{V}_t^0(s_t^k)$. In this case, the induction hypothesis implies that $\forall s', \bar{V}_{t+1}^{k-1}(s') \leq \bar{V}_{t+1}^0(s')$ and the result can be proven similarly to the base case. □

Lemma 2 (Value Update for Exact Model)**.** *The expected cumulative value update of RTDP at the k^{th} episode satisfies*

$$\bar{V}_1^{k-1}(s_1^k) - V_1^{\pi_k}(s_1^k) = \sum_{t=1}^{H} \mathbb{E}[\bar{V}_t^{k-1}(s_t^k) - \bar{V}_t^k(s_t^k) \mid \mathcal{F}_{k-1}].$$

Proof. By the definition of a_t^k and the update rule, the following holds:

$$\begin{aligned}\mathbb{E}\big[\bar{V}_t^k(s_t^k) \mid \mathcal{F}_{k-1}\big] &= \mathbb{E}\big[r(s_t^k, a_t^k) + p(\cdot \mid s_t^k, a_t^k)^T \bar{V}_{t+1}^{k-1} \mid \mathcal{F}_{k-1}\big] \\ &= \mathbb{E}\big[r(s_t^k, a_t^k) \mid \mathcal{F}_{k-1}\big] + \mathbb{E}\left[\sum_{\bar{s}_{t+1}} p(\bar{s}_{t+1} \mid s_t, \pi_k)\bar{V}_{t+1}^{k-1}(\bar{s}_{t+1}) \mid \mathcal{F}_{k-1}\right].\end{aligned}$$

Furthermore,

$$\begin{aligned}&\mathbb{E}\left[\sum_{\bar{s}_{t+1}} p(\bar{s}_{t+1} \mid s_t^k, \pi_k)\bar{V}_{t+1}^{k-1}(\bar{s}_{t+1}) \mid \mathcal{F}_{k-1}\right] \\ &= \sum_{s_t^k} \Pr(s_t^k \mid s_1^k, \pi_k) \sum_{\bar{s}_{t+1} \in \mathcal{S}} p(\bar{s}_{t+1} \mid s_t^k, \pi_k)\bar{V}_{t+1}^{k-1}(\bar{s}_{t+1}) \\ &= \sum_{s_{t+1}^k \in \mathcal{S}} \Pr(s_{t+1}^k \mid s_1^k, \pi_k)\bar{V}_{t+1}^{k-1}(s_{t+1}^k) = \mathbb{E}\big[\bar{V}_{t+1}^{k-1}(s_{t+1}^k) \mid \mathcal{F}_{k-1}\big]. \end{aligned} \tag{8}$$

The first relation holds by definition and the second one holds by the Markovian property of the dynamics. Substituting back and summing both side from $t = 1, \ldots, H$, we obtain

$$\begin{aligned}\mathbb{E}\left[\sum_{t=1}^{H} \bar{V}_t^k(s_t^k) \mid \mathcal{F}_{k-1}\right] &= \mathbb{E}\left[\sum_{t=1}^{H} r(s_t^k, a_t^k) \mid \mathcal{F}_{k-1}\right] + \mathbb{E}\left[\sum_{t=1}^{H} \bar{V}_{t+1}^{k-1}(s_{t+1}^k) \mid \mathcal{F}_{k-1}\right] \\ &= \mathbb{E}\left[\sum_{t=1}^{H} r(s_t^k, a_t^k) \mid \mathcal{F}_{k-1}\right] + \mathbb{E}\left[\sum_{t=1}^{H} \bar{V}_t^{k-1}(s_t^k) \mid \mathcal{F}_{k-1}\right] - \bar{V}_1^{k-1}(s_1^k) \\ &= V_1^{\pi_k}(s_1^k) + \mathbb{E}\left[\sum_{t=1}^{H} \bar{V}_t^{k-1}(s_t^k) \mid \mathcal{F}_{k-1}\right] - \bar{V}_1^{k-1}(s_1^k)\end{aligned}$$

The second line hold by shifting the index of the sum and using the fact that $\forall s,\ \bar{V}_{H+1}^k(s) = 0$. The third line holds by the definition of the value function,

$$\sum_{t=1}^{H} \mathbb{E}[r(s_t^k, a_t^k) \mid \mathcal{F}_{k-1}] = \mathbb{E}[\sum_{t=1}^{H} r(s_t^k, a_t^k) \mid s_1 = s_1^k] = V_1^{\pi_k}(s_1^k).$$

Reorganizing the equation yields the desired result. □

Theorem 4 (Regret Bounds for RTDP)**.** *The following regret bounds hold for RTDP:*

1. $\mathbb{E}[\mathrm{Regret}(K)] \leq SH^2$.
2. *For any* $\delta > 0$*, with probability* $1-\delta$*, for all* $K > 0$*,* $\mathrm{Regret}(K) \leq 9SH^2 \ln(3SH/\delta)$.

Proof. The following bounds on the regret hold.

$$\begin{aligned}\mathrm{Regret}(K) := \sum_{k=1}^{K} V^*(s_1^k) - V^{\pi_k}(s_1^k) &\leq \sum_{k=1}^{K} \bar{V}_1^{k-1}(s_1^k) - V^{\pi_k}(s_1^k) \\ &\leq \sum_{k=1}^{K}\sum_{t=1}^{H} \mathbb{E}[\bar{V}_t^{k-1}(s_t^k) - \bar{V}_t^k(s_t^k) \mid \mathcal{F}_{k-1}]. \qquad (9)\end{aligned}$$

The second relation is by the optimism of the value function (Lemma 1), and the third relation is by Lemma 2.

To prove the bound on the expected regret, we take expectation on both sides of (9). Thus,

$$\mathbb{E}[\mathrm{Regret}(K)] \leq \sum_{k=1}^{K} \mathbb{E}[\mathbb{E}[\sum_{t=1}^{H} \bar{V}_t^{k-1}(s_t^k) - \bar{V}_t^k(s_t^k) \mid \mathcal{F}_{k-1}]] = \mathbb{E}[\sum_{k=1}^{K}\sum_{t=1}^{H} \bar{V}_t^{k-1}(s_t^k) - \bar{V}_t^k(s_t^k)].$$

Where the second relation holds by the tower property and linearity of expectation. Finally, for any run of RTDP, we have that

$$\sum_{k=1}^{K}\sum_{t=1}^{H} \bar{V}_t^{k-1}(s_t^k) - \bar{V}_t^k(s_t^k) = \sum_{s}\sum_{t=1}^{H} \bar{V}_t^0(s) - \bar{V}_t^K(s) \leq \sum_{s}\sum_{t=1}^{H} \bar{V}_t^0(s) - V_t^*(s) \leq SH^2.$$

The first relation holds since per s, the sum is telescopic, thus, only the first and last term exist in the sum. Due to the update rule, on the first time a state appears, its value will be $\bar{V}_t^0(s)$. From the last time it appears, its value will not be updated and thus the last value of a state is $\bar{V}_t^K(s)$. The second relation holds by Lemma 1. The third relation holds since $\forall s,t,\ \bar{V}_t^0(s) - V_t^*(s) \in [0,H]$, summing on SH such terms yields the result.

To prove the high-probability bound we apply Lemma 34 by which,

$$(9) = \sum_{t=1}^{H}\sum_{s}\sum_{k=1}^{K} \bar{V}_t^{k-1}(s) - \mathbb{E}[\bar{V}_t^k(s) \mid \mathcal{F}_{k-1}].$$

For a fixed s,t, $\left\{\bar{V}_t^k(s)\right\}_{k\geq 0}$ is a Decreasing Bounded Process by Lemma 1, and its initial value is less than H. Thus, (9) is a sum of SH Decreasing Bounded Processes. We apply Lemma 11 which provides a high-probability bound on a sum of Decreasing Bounded Processes to conclude the proof. □

Corollary 5 (RTDP is Uniform PAC)**.** *Let $\delta > 0$ and N_ϵ be the number of episodes in which RTDP outputs a policy with $V_1^*(s_1^k) - V_1^{\pi_k}(s_1^k) > \epsilon$. Then,*

$$\Pr\left\{\exists \epsilon > 0 : N_\epsilon \geq \frac{9SH^2 \ln(3SH/\delta)}{\epsilon}\right\} \leq \delta.$$

Proof. Let K_{N_ϵ} be an episode index such that there are N_ϵ previous episodes $k \leq K_{N_\epsilon}$ in which RTDP outputs a policy with $V_1^*(s_1^k) - V_1^{\pi_k}(s_1^k) > \epsilon$. The following relation holds,

$$\forall \epsilon > 0 : kkvbvuvdhfirinhblbkudchurbknbulrN_\epsilon\epsilon \leq \text{Regret}(K_{N_\epsilon}).$$

Thus,

$$\begin{aligned}\left\{\exists \epsilon > 0 : N_\epsilon \epsilon \geq 9SH^2 \ln \frac{3SH}{\delta}\right\} &\subseteq \left\{\text{Regret}(K_{N_\epsilon}) \geq 9SH^2 \ln \frac{3SH}{\delta}\right\} \\ &\subseteq \left\{\exists K > 0 : \text{Regret}(K) \geq 9SH^2 \ln \frac{3SH}{\delta}\right\}.\end{aligned}$$

Which results in

$$\Pr\left\{\exists \epsilon > 0 : N_\epsilon \epsilon \geq 9SH^2 \ln \frac{3SH}{\delta}\right\} \leq \Pr\left\{\exists K > 0 : \text{Regret}(K) \geq 9SH^2 \ln \frac{3SH}{\delta}\right\} \leq \delta.$$

where the third relation holds by Theorem 4. □

Corollary 6 (RTDP is $(0, \delta)$ PAC)**.** *Let $\delta > 0$ and N be the number of episodes in which RTDP outputs a non optimal policy. Define the (unknown) gap of the MDP, $\Delta(\mathcal{M}) = \min_s \min_{\pi : V_1^\pi(s) \neq V_1^*(s)} V_1^*(s) - V_1^\pi(s) > 0$. Then,*

$$\Pr\left\{N \geq \frac{9SH^2 \ln(3SH/\delta)}{\Delta(\mathcal{M})}\right\} \leq \delta.$$

Proof. We have that $N = N_{\Delta(\mathcal{M})}$ since $\Delta(\mathcal{M})$ is the minimal gap; in all rest of episodes in which the gap is smaller than $\Delta(\mathcal{M})$, the policy π_k is necessarily the optimal one. Based on Corollary 5 we conclude that,

$$\Pr\left\{N \geq \frac{9SH^2 \ln \frac{3SH}{\delta}}{\Delta(\mathcal{M})}\right\} = \Pr\left\{N_{\Delta(\mathcal{M})} \geq \frac{9SH^2 \ln \frac{3SH}{\delta}}{\Delta(\mathcal{M})}\right\} \leq \delta.$$

□

C Proofs of Section 4

Lemma 7 (Value Update for Optimistic Model)**.** *The expected cumulative value update of Algorithm 2 in the k^{th} episode is bounded by*

$$\bar{V}_1^{k-1}(s_1^k) - V_1^{\pi_k}(s_1^k) \leq \sum_{t=1}^{H} \mathbb{E}\big[\bar{V}_t^{k-1}(s_t^k) - \bar{V}_t^k(s_t^k) \mid \mathcal{F}_{k-1}\big] + \sum_{t=1}^{H} \mathbb{E}\big[(\tilde{r}_{k-1} - r)(s_t^k, a_t^k) + (\tilde{p}_{k-1} - p)(\cdot \mid s_t^k, a_t^k)^T \bar{V}_{t+1}^{k-1} \mid \mathcal{F}_{k-1}\big] \ .$$

We prove a more general, Lemma 12, of which Lemma 7 is a direct corollay (by setting $t = 1$).

Lemma 12. *The expected value update of Algorithm 2 in the k^{th} episode at the state t^{th} is bounded by*

$$\bar{V}_t^{k-1}(s_t^k) - V_t^{\pi_k}(s_t^k) \leq \sum_{t'=t}^{H} \mathbb{E}\big[\bar{V}_{t'}^{k-1}(s_{t'}^k) - \bar{V}_{t'}^k(s_{t'}^k) \mid \mathcal{F}_{k-1}, s_t^k\big] + \sum_{t'=t}^{H} \mathbb{E}\big[(\tilde{r}_{k-1} - r)(s_{t'}^k, a_{t'}^k) + (\tilde{p}_{k-1} - p)(\cdot \mid s_{t'}^k, a_{t'}^k)^T \bar{V}_{t'+1}^{k-1} \mid \mathcal{F}_{k-1}, s_t^k\big] \ .$$

Proof. We closely follow the proof of Lemma 2. By the definition of a_t^k and the update rule, for $t' \geq t$, the following holds.

$$\begin{aligned}
&\mathbb{E}\big[\bar{V}_{t'}^k(s_{t'}^k) \mid \mathcal{F}_{k-1}, s_t^k\big] \\
&\overset{(1)}{\leq} \mathbb{E}\left[\tilde{r}_{k-1}(s_{t'}^k, a_{t'}^k) + \sum_{\bar{s}_{t'+1} \in \mathcal{S}} \tilde{p}_{k-1}(\bar{s}_{t'+1} \mid s_{t'}^k, a_{t'}) \bar{V}_{t'+1}^{k-1}(\bar{s}_{t'+1}) \mid \mathcal{F}_{k-1}, s_t^k\right] \\
&\overset{(2)}{=} \mathbb{E}\big[r(s_{t'}^k, a_{t'}^k) \mid \mathcal{F}_{k-1}, s_{t'}^k\big] + \mathbb{E}\left[\sum_{\bar{s}_{t'+1} \in \mathcal{S}} p(\bar{s}_{t'+1} \mid s_{t'}^k, a_{t'}^k) \bar{V}_{t'+1}^{k-1}(\bar{s}_{t'+1}) \mid \mathcal{F}_{k-1}, s_t^k\right] \\
&\quad + \mathbb{E}\left[(\tilde{r}_{k-1} - r)(s_{t'}^k, a_{t'}^k) + \sum_{\bar{s}_{t'+1} \in \mathcal{S}} (\tilde{p}_{k-1} - p)(\bar{s}_{t'+1} \mid s_{t'}^k, a_{t'}^k) \bar{V}_{t'+1}^{k-1}(\bar{s}_{t'+1}) \mid \mathcal{F}_{k-1}, s_t^k\right] \\
&\overset{(3)}{=} \mathbb{E}\big[r(s_{t'}^k, a_{t'}^k) \mid \mathcal{F}_{k-1}, , s_t^k\big] + \mathbb{E}\big[\bar{V}_{t'+1}^{k-1}(s_{t'+1}^k) \mid \mathcal{F}_{k-1}, s_t^k\big] \\
&\quad + \mathbb{E}\left[(\tilde{r}_{k-1} - r)(s_{t'}^k, a_{t'}^k) + \sum_{\bar{s}_{t'+1} \in \mathcal{S}} (\tilde{p}_{k-1} - p)(\bar{s}_{t'+1} \mid s_{t'}^k, a_{t'}^k) \bar{V}_{t'+1}^{k-1}(\bar{s}_{t'+1}) \mid \mathcal{F}_{k-1}, s_t^k\right]
\end{aligned}$$

Relation (1) holds by the update rule for $\bar{V}_t^k$. Next, (2) holds by adding an subtracting the real reward and dynamics and using linearity of expectation. In (3), we used the same reasoning as in Equation (8).

Summing both side from $t' = t, \dots, H$, we obtain:

$$
\begin{aligned}
&\mathbb{E}\left[\sum_{t'=t}^{H} \bar{V}_{t'}^{k}(s_{t'}^{k}) \mid \mathcal{F}_{k-1}, s_t^k\right] \\
&\leq \mathbb{E}\left[\sum_{t'=t}^{H} r(s_{t'}^{k}, a_{t'}^{k}) \mid \mathcal{F}_{k-1}, s_t^k\right] + \mathbb{E}\left[\sum_{t'=t}^{H} \bar{V}_{t'+1}^{k-1}(s_{t'+1}^{k}) \mid \mathcal{F}_{k-1}, s_t^k\right] \\
&\quad + \sum_{t'=t}^{H} \mathbb{E}\left[(\tilde{r}_{k-1} - r)(s_{t'}^{k}, a_{t'}^{k}) + \sum_{\bar{s}_{t'+1} \in \mathcal{S}} (\tilde{p}_{k-1} - p)(\bar{s}_{t'+1} \mid s_{t'}^{k}, a_{t'}^{k}) \bar{V}_{t'+1}^{k-1}(\bar{s}_{t'+1}) \mid \mathcal{F}_{k-1}, s_t^k\right] \\
&\overset{(1)}{=} V_t^{\pi_k}(s_t^k) + E\left[\sum_{t'=t}^{H} \bar{V}_{t'+1}^{k-1}(s_{t'+1}^{k}) \mid \mathcal{F}_{k-1}, s_t^k\right] \\
&\quad + \sum_{t'=t}^{H} \mathbb{E}\left[(\tilde{r}_{k-1} - r)(s_{t'}^{k}, a_{t'}^{k}) + \sum_{\bar{s}_{t'+1} \in \mathcal{S}} (\tilde{p}_{k-1} - p)(\bar{s}_{t'+1} \mid s_{t'}^{k}, a_{t'}^{k}) \bar{V}_{t'+1}^{k-1}(\bar{s}_{t'+1}) \mid \mathcal{F}_{k-1}, s_t^k\right] \\
&\overset{(2)}{=} V_t^{\pi_k}(s_t^k) + \mathbb{E}\left[\sum_{t'=t}^{H} \bar{V}_{t'}^{k-1}(s_{t'}^{k}) \mid \mathcal{F}_{k-1}, s_t^k\right] - \bar{V}_t^{k-1}(s_t^k) \\
&\quad + \sum_{t'=t}^{H} \mathbb{E}\left[(\tilde{r}_{k-1} - r)(s_{t'}^{k}, a_{t'}^{k}) + \sum_{\bar{s}_{t'+1} \in \mathcal{S}} (\tilde{p}_{k-1} - p)(\bar{s}_{t'+1} \mid s_{t'}^{k}, a_{t'}^{k}) \bar{V}_{t'+1}^{k-1}(\bar{s}_{t'+1}) \mid \mathcal{F}_{k-1}, s_t^k\right]
\end{aligned}
$$

In (1) we used the fact that $V_t^{\pi_k}(s_t^k) = \mathbb{E}[\sum_{t'=t}^{H} r(s_{t'}^k, \pi_k(s_{t'}^k)) \mid \mathcal{F}_{k-1}, s_t^k]$. Relation (2) holds by shifting the index of the sum and using $\forall s, k,\ \bar{V}_{H+1}^{k-1}(s) = 0$. Reorganizing the equation yields the desired result. □

D Proof of Theorem 8

Algorithm 4 UCRL2 with Greedy Policies

Initialize: $\delta, \delta' = \frac{\delta}{4} \forall s \in \mathcal{S}, t \in [H],\ \bar{V}_t^0(s) = H - (t-1)$.
for $k = 1, 2, ..$ **do**
  Initialize s_1^k
  for $t = 1, .., H$ **do**
    #Update Upper Bound on V^*
    for $a \in \mathcal{A}$ **do**
      $\tilde{r}_{k-1}(s_t^k, a) = \hat{r}_{k-1}(s_t^k, a) + \sqrt{\frac{2 \ln \frac{2SAT}{\delta'}}{n_{k-1}(s_t^k, a) \vee 1}}$
      $CI(s_t^k, a) = \left\{ P' \in \mathcal{P}(\mathcal{S}) : \|P'(\cdot) - \hat{p}_{k-1}(\cdot \mid s_t^k, a)\|_1 \leq \sqrt{\frac{4S \ln \frac{3SAT}{\delta'}}{n_{k-1}(s_t^k, a) \vee 1}} \right\}$
      $\tilde{p}_{k-1}(\cdot \mid s_t^k, a) = \arg\max_{P' \in CI(s_t^k, a)} P'(\cdot \mid s_t^k, a)^T \bar{V}_{t+1}^{k-1}$
      $\bar{Q}(s_t^k, a) = \tilde{r}_{k-1}(s_t^k, a) + \tilde{p}_{k-1}(\cdot \mid s_t^k, a)^T \bar{V}_{t+1}^{k-1}$
    end for
    $a_t^k \in \arg\max_a \bar{Q}(s_t^k, a)$
    $\bar{V}_t^k(s_t^k) = \min\{\bar{V}_t^{k-1}(s_1^k), \bar{Q}(s_t^k, a_t^k)\}$
    #Act by the Greedy Policy
    Apply a_t^k and observe s_{t+1}^k.
  end for
  Update $\hat{r}_k, \hat{p}_k, n_k$ with all experience gathered in the episode.
end for

We provide the full proof of Theorem 8 which establishes a regret bound for UCRL2 with Greedy Policies (UCRL2-GP) in finite horizon MDPs. In the following, we present the structure of this section.

We define the failure events for UCRL2-GP in Section D.1. Most of the events are standard low-probability failure events, derived using, e.g., Hoeffding's inequality. We add to the standard set of events a failure event which holds when a sum of Decreasing Bounded Processes is large in its value. Using uniform bounds, the failure events are shown to hold jointly. When all failure events do not occur for all time-steps we say the algorithm is outside the failure event. In Section D.2 we establish that UCRL2-GP is optimistic, and, more specifically, that for all $s, t, k\ \bar{V}_t^k(s) \geq V_t^*(s)$, outside the failure event. Lastly, in Section D.3 we give the full proof of Theorem 8, based on a new regret decomposition using on Lemma 7, the new results on Decreasing Bounded Processes (see Appendix A), and existing techniques (e.g., [Dann et al., 2017, Zanette and Brunskill, 2019]).

D.1 Failure Events for UCRL2 with Greedy Policies

Define the following failure events.

$$F_k^r = \left\{ \exists s, a :\ |r(s,a) - \hat{r}_{k-1}(s,a)| \geq \sqrt{\frac{2 \ln \frac{2SAT}{\delta'}}{n_{k-1}(s,a) \vee 1}} \right\}$$

$$F_k^p = \left\{ \exists s, a :\ \|p(\cdot \mid s,a) - \hat{p}_{k-1}(\cdot \mid s,a)\|_1 \geq \sqrt{\frac{4S \ln \frac{3SAT}{\delta'}}{n_{k-1}(s,a) \vee 1}} \right\}$$

$$F_k^N = \left\{ \exists s, a : n_{k-1}(s,a) \leq \frac{1}{2} \sum_{j<k} w_j(s,a) - H \ln \frac{SAH}{\delta'} \right\}.$$

$$F^{DBP} = \left\{ \exists k > 0 : \sum_{k=1}^{K} \sum_{t=1}^{H} \sum_{s} \bar{V}_t^{k-1}(s) - \mathbb{E}[\bar{V}_t^k(s) \mid \mathcal{F}_{k-1}] \geq 9SH^2 \ln \frac{3SH}{\delta'} \right\}$$

Furthermore, the following relations hold.

- Let $F^r = \bigcup_{k=1}^{K} F_k^r$. Then $\Pr\{F^r\} \leq \delta'$, by Hoeffding's inequality, and using a union bound argument on all s, a, possible values of $n_k(s,a)$ and k. Furthermore, for $n(s,a) = 0$ the bound holds trivially since $R \in [0,1]$.
- Let $F^P = \bigcup_{k=1}^{K} F_k^p$. Then $\Pr\{F^p\} \leq \delta'$, holds by [Weissman et al., 2003] while applying union bound on all $s, a, n_{k-1}(s,a)$ and possible values of k (e.g., Azar et al. 2017, Zanette and Brunskill 2019). Furthermore, for $n(s,a) = 0$ the bound holds trivially.
- Let $F^N = \bigcup_{k=1}^{K} F_k^N$. Then, $\Pr\{F^N\} \leq \delta'$. The proof is given in [Dann et al., 2017] Corollary E.4 (and is used in Zanette and Brunskill 2019 Appendix D.4).
- By construction of Algorithm 2, $\forall s, t,\ \bar{V}_t^k(s)$ is a decreasing function of k, with $\bar{V}_t^0(s) = H$. Furthermore, since $\hat{r}_{k-1}(s,a)$ and $\hat{p}_{k-1}(\cdot \mid s,a)$ are non-negative, and $\bar{V}_t^0(s) > 0$, a simple induction allows us to conclude that $\forall s, t,\ \bar{V}_t^k(s) \geq 0$. Thus, by applying Lemma 11, $\Pr\{F^{DBP}\} \leq \delta'$.

Lemma 13. *Setting $\delta' = \frac{\delta}{4}$ then $\Pr\{F^r \bigcup F^p \bigcup F^N \bigcup F^{DBP}\} \leq \delta$. When the failure events does not hold we say the algorithm is outside the failure event.*

D.2 UCRL2 with Greedy Policies is Optimistic

Lemma 14. *Outside the failure event UCRL2-GP is Optimistic,*

$$\forall s, t, k\ \bar{V}_t^k(s) \geq V_t^*(s).$$

Proof. We prove by induction. The base case holds by the initialization of the algorithm, $\bar{V}_t^0(s) = H - (t-1) \geq V_t^*(s)$. Assume the induction hypothesis holds for $k-1$ episodes. At the k^{th} episode, states there were not visited at step t will not be updated, and thus by the induction hypothesis, the result hold for these states. For states that were visited, if the minimum at the update stage equals to $\bar{V}_t^{k-1}(s)$, then the result similarly holds. Let s_t^k be a state that was updated according to the optimistic model, and let

$$\tilde{a}^* \in \arg\max_a \tilde{r}_{k-1}(s_t^k, a) + \tilde{p}_{k-1}(\cdot \mid s_t^k, a) v_{t+1}^{k-1}$$
$$a^* \in \arg\max_a r(s_t^k, a) + p(\cdot \mid s_t^k, a) V_{t+1}^*.$$

Then,

$$\begin{aligned}
\bar{V}_t^k(s_t^k) &= \max_a \tilde{r}_{k-1}(s_t^k, a) + \tilde{p}_{k-1}(\cdot \mid s_t^k, a) \bar{V}_{t+1}^{k-1} \\
&\overset{(1)}{=} \tilde{r}_{k-1}(s_t^k, \tilde{a}^*) + \tilde{p}_{k-1}(\cdot \mid s_t^k, \tilde{a}^*) \bar{V}_{t+1}^{k-1} \\
&\overset{(2)}{\geq} \tilde{r}_{k-1}(s_t^k, a^*) + \tilde{p}_{k-1}(\cdot \mid s_t^k, a^*) \bar{V}_{t+1}^{k-1} \\
&\overset{(3)}{\geq} r(s_t^k, a^*) + p(\cdot \mid s_t^k, a^*) \bar{V}_{t+1}^{k-1} \\
&\overset{(4)}{\geq} r(s_t^k, a^*) + p(\cdot \mid s_t^k, a^*) V_{t+1}^* \\
&\overset{(5)}{=} V_t^*(s_t^k).
\end{aligned}$$

Relations (1) and (2) are by the definition and optimality of $\tilde{a}^*$, respectively. (3) holds since outside failure event F_k^r, $\tilde{r}_{k-1}(s_t^k, a^*) \geq r(s_t^k, a^*)$. Furthermore, outside failure event F_k^p, the real transition probabilities $p(\cdot \mid s, a^*) \in CI(s_t^k, a^*)$, and thus

$$\max_{P' \in CI(s_t^k, a^*)} P'(\cdot \mid s_t^k, a^*) \bar{V}_{t+1}^{k-1} = \tilde{p}_{k-1}(\cdot \mid s_t^k, a^*) \bar{V}_{t+1}^{k-1} \geq p_{k-1}(\cdot \mid s_t^k, a^*) \bar{V}_{t+1}^{k-1}.$$

Finally, (4) holds by the induction hypothesis $\forall s, a, t,\ \bar{V}_t^{k-1}(s) \geq V_t^*(s)$ and (5) holds by the Bellman recursion. □

D.3 Proof of Theorem 8

Theorem 8 (Regret Bound of UCRL2-GP)**.** *For any time $T \leq KH$, with probability at least $1-\delta$, the regret of UCRL2-GP is bounded by $\tilde{\mathcal{O}}\Big(HS\sqrt{AT} + H^2\sqrt{S}SA\Big)$.*

Proof. By the optimism of the value (Lemma 14), we have that

$$\begin{aligned}\sum_{k=1}^{K} V_1^*(s_1^k) - V_1^{\pi_k}(s_1^k) &\leq \sum_{k=1}^{K} \bar{V}_1^{k-1}(s_1^k) - V_1^{\pi_k}(s_1^k) \\ &\leq \underbrace{\sum_{k=1}^{K}\sum_{t=1}^{H} \mathbb{E}[\bar{V}_t^{k-1}(s_t^k) - \bar{V}_t^k(s_t^k) \mid \mathcal{F}_{k-1}]}_{(A)} \\ &\quad + \underbrace{\sum_{k=1}^{K}\sum_{t=1}^{H} \mathbb{E}\left[(\tilde{r}_{k-1} - r)(s_t^k, a_t^k) + (\tilde{p}_{k-1} - p)(\cdot \mid s_t^k, a_t^k)^T \bar{V}_{t+1}^{k-1} \mid \mathcal{F}_{k-1}\right]}_{(B)} . \quad (10)\end{aligned}$$

The first relation is by the optimism of the value, and the second relation is by Lemma 7. We now bound the two terms outside the failure event.

Bounding (A). By Lemma 34 (Appendix F),

$$(A) = \sum_{k=1}^{K}\sum_{t=1}^{H}\sum_{s} \bar{V}_t^{k-1}(s) - \mathbb{E}[\bar{V}_t^k(s) \mid \mathcal{F}_{k-1}].$$

Outside the failure event, the sum is bounded by $9SH^2 \ln \frac{3SH}{\delta'}$ (see event F^{DBP}). Thus,

$$(A) \leq \tilde{\mathcal{O}}(SH^2).$$

Bounding (B). Outside failure event F_k^r the following inequality holds:

$$\begin{aligned}&\sum_{k=1}^{K}\sum_{t=1}^{H} \mathbb{E}\left[(\tilde{r}_{k-1} - r)(s_t^k, \pi_k(s_t^k) \mid \mathcal{F}_{k-1}\right] \\ &\lesssim \sum_{k=1}^{K}\sum_{t=1}^{H} \mathbb{E}\left[\sqrt{\frac{1}{n_{k-1}(s_t^k, \pi_k(s_t^k)) \vee 1}} \mid \mathcal{F}_{k-1}\right] \lesssim \tilde{\mathcal{O}}(\sqrt{SAT} + SAH), \quad (11)\end{aligned}$$

where the second inequality is by Lemma 38. It is worth noting that Lemma 38 is proven by defining L_k, the set of 'good' state-action pairs, that contains pairs that were visited sufficiently often in the past [Dann et al., 2017, Zanette and Brunskill, 2019]. The term we bound is then analyzed separately for state-action pairs inside and outside L_k. The definition of L_k can be found in Definition 2, and its properties (including Lemma) are analyzed in Appendix F.1.

Furthermore, outside the failure event,

$$
\begin{aligned}
&\sum_{k=1}^{K}\sum_{t=1}^{H}\mathbb{E}\left[(\tilde{p}_{k-1}-p)(\cdot \mid s_t^k,a_t^k)^T\bar{V}_{t+1}^{k-1} \mid \mathcal{F}_{k-1}\right] \\
&=\sum_{k=1}^{K}\sum_{t=1}^{H}\mathbb{E}\left[(\tilde{p}_{k-1}-\hat{p}_{k-1})(\cdot \mid s_t^k,a_t^k)^T\bar{V}_{t+1}^{k-1} \mid \mathcal{F}_{k-1}\right] \\
&\qquad +\mathbb{E}\left[(\hat{p}_{k-1}-p)(\cdot \mid s_t^k,a_t^k)^T\bar{V}_{t+1}^{k-1} \mid \mathcal{F}_{k-1}\right] \\
&\overset{(1)}{\leq}\sum_{k=1}^{K}\sum_{t=1}^{H}\mathbb{E}\left[\|(\tilde{p}_{k-1}-\hat{p}_{k-1})(\cdot \mid s_t^k,a_t^k)\|_1\|\bar{V}_{t+1}^{k-1}\|_\infty \mid \mathcal{F}_{k-1}\right] \\
&\qquad +\mathbb{E}\left[\|(\hat{p}_{k-1}-p)(\cdot \mid s_t^k,a_t^k)\|\|\bar{V}_{t+1}^{k-1}\|_\infty \mid \mathcal{F}_{k-1}\right] \\
&\overset{(2)}{\leq} H\sum_{k=1}^{K}\sum_{t=1}^{H}\mathbb{E}\left[\|(\hat{p}_{k-1}-p)(\cdot \mid s_t^k,a_t^k)\|_1+\|(\tilde{p}_{k-1}-\hat{p}_{k-1})(\cdot \mid s_t^k,a_t^k)\|_1 \mid \mathcal{F}_{k-1}\right] \\
&\overset{(3)}{\lesssim} H\sqrt{S}\sum_{k=1}^{K}\sum_{t=1}^{H}\mathbb{E}\left[\sqrt{\frac{1}{n_{k-1}(s_t^k,a_t^k)\vee 1}} \mid \mathcal{F}_{k-1}\right] \\
&\overset{(4)}{\lesssim}\tilde{\mathcal{O}}(HS\sqrt{AT}+H^2\sqrt{S}SA). \qquad (12)
\end{aligned}
$$

Relation (1) holds by Hölder's inequality. Next, (2) holds since $\forall s,t,k,\ 0\leq\bar{V}_t^k(s)\leq H$. The lower bounds holds by Lemma 14 and since $V_t^*\geq 0$. The upper bound is since the value can only decrease by Algorithm 2 and the inequality holds for the initialized value. Finally, (3) holds outside failure event F^p (Lemma 13), and (4) holds by Lemma 38.

Combining (11), (12) we conclude that,

$$(B)\leq\tilde{\mathcal{O}}(HS\sqrt{AT}+H^2\sqrt{S}SA)$$

Combining the bounds on (A) and (B) in (10) concludes the proof. □

E Proof of Theorem 9

Algorithm 5 EULER with Greedy Policies

Initialize: $\delta, \delta' = \frac{\delta}{9}, \forall s \in \mathcal{S}, t \in [H],\ \bar{V}_t^0(s) = H - (t-1), \underline{V}_t^0(s) = 0,$

$$\phi(s,a) = \sqrt{\frac{2\hat{\mathrm{Var}}_{\hat{p}_{k-1}(s,a)}(\bar{V}_{t+1}^{k-1}) \ln \frac{4SAT}{\delta'}}{n_{k-1}(s,a)}} + \frac{2H \ln \frac{4SAT}{\delta'}}{3n_{k-1}(s,a)}, L = 2\sqrt{\ln \frac{4SAT}{\delta'}},$$

$$J = \frac{2H \ln \frac{4SAT}{\delta'}}{3}, B_v = \sqrt{2 \ln \frac{4SAT}{\delta'}}, B_p = H\sqrt{2 \ln \frac{4SAT}{\delta'}}.$$

for $k = 1, 2, ..$ **do**
  Initialize s_1^k
  for $t = 1, .., H$ **do**
    #Update Upper Bound on V^*
    for $a \in \mathcal{A}$ **do**

$$b_k^r(s_t^k, a) = \sqrt{\frac{2\hat{\mathrm{Var}}(R(s_t^k,a)) \ln \frac{4SAT}{\delta'}}{n_{k-1}(s_t^k,a) \vee 1}} + \frac{14 \ln \frac{4SAT}{\delta'}}{3n_{k-1}(s_t^k,a) \vee 1}$$

$$b_k^{pv}(s_t^k, a) = \phi(\hat{p}_{k-1}(\cdot \mid s_t^k, a), \bar{V}_{t+1}^{k-1}) + \frac{4J + B_p}{n_{k-1}(s_t^k,a) \vee 1} + \frac{B_v \|\bar{V}_{t+1}^{k-1} - \underline{V}_{t+1}^{k-1}\|_{2,\hat{p}}}{\sqrt{n_{k-1}(s_t^k,a) \vee 1}}$$

$$\bar{Q}(s_t^k, a) = \hat{r}_{k-1}(s_t^k, a) + b_k^r(s_t^k, a) + \hat{p}_{k-1}(\cdot \mid s_t^k, a)^T \bar{V}_{t+1}^{k-1} + b_k^{pv}(s_t^k, a)$$

    end for
    $a_t^k \in \arg\max_a \bar{Q}(s_t^k, a)$
    $\bar{V}_t^k(s_t^k) = \min\{\bar{V}_t^{k-1}(s_t^k), \bar{Q}(s_t^k, a_t^k)\}$
    #Update Lower Bound on V^*

$$b_k^{pv}(s_t^k, a_t^k) = \phi(\hat{p}_{k-1}(\cdot \mid s_t^k, a_t^k), \underline{V}_{t+1}^{k-1}) + \frac{4J + B_p}{n_{k-1}(s_t^k,a_t^k) \vee 1} + \frac{B_v \|\bar{V}_{t+1}^{k-1} - \underline{V}_{t+1}^{k-1}\|_{2,\hat{p}}}{\sqrt{n_{k-1}(s_t^k,a_t^k) \vee 1}}$$

$$\underline{Q}(s_t^k, a_t^k) = \hat{r}_{k-1}(s_t^k, a_t^k) - b_k^r(s_t^k, a_t^k) + \hat{p}_{k-1}(\cdot \mid s_t^k, a)^T \underline{V}_{t+1}^{k-1} - b_k^{pv}(s_t^k, a_t^k)$$

$$\underline{V}_t^k(s_t^k) = \max\left\{\underline{V}_t^{k-1}(s_t^k), \underline{Q}(s_t^k, a_t^k)\right\}$$

    #Act by the Greedy Policy
    Apply a_t^k and observe s_{t+1}^k.
  end for
  Update $\hat{r}_k, \hat{p}_k, n_k$ with all experience gathered in episode.
end for

Remark 1. *Note that the algorithm does not explicitly define $\tilde{r}_{k-1}(s,a)$ and $\tilde{p}_{k-1}(\cdot \mid s,a)$. While we can directly set $\tilde{r}_{k-1}(s,a) = \hat{r}_{k-1}(s,a) + b_k^r(s_t^k, a)$, the optimistic transition kernel is only implicitly defined as*

$$\tilde{p}_{k-1}(\cdot \mid s,a)^T \bar{V}_t^{k-1} = \hat{p}_{k-1}(\cdot \mid s,a)^T \bar{V}_{t+1}^{k-1} + b_k^{pv}(s,a)$$

Nevertheless, throughout the proofs we are only interested in the above quantity, and thus, except for some abuse of notation, all of the proofs hold. We use this notation since it is common in previous works (e.g., Zanette and Brunskill 2019, Dann et al. 2017) and for brevity.

In this section, we provide the full proof of Theorem 9 which establishes a regret bound for EULER with Greedy Policies (EULER-GP). In Zanette and Brunskill [2019] the authors prove their results using a general confidence interval, which they refer as *admissible confidence interval*. In Section E.1 we state there definition and state some useful properties. In Section E.2 we define the set of failure events and show that with high-probability the failure events do not occur. The set of failure events includes high-probability events derived using empirical Bernstein inequalities [Maurer and Pontil, 2009], as well as high probability events on Decreasing Bounded Process, as we establish in Appendix A. In Section E.3 we analyze the optimism EULER-GP and prove it satisfies the same optimism and pessimism as in Zanette and Brunskill [2019], outside the failure event for all s, t, k $\underline{V}_t^k(s) \leq V_t^*(s) \leq \bar{V}_t^k(s)$.

In Section E.4, using the above, we give the full proof of Theorem 9. As for the proof of UCRL2-GP, we apply the new suggested regret decomposition, based on Lemma 7, and use the new results on Decreasing Bounded Processes. In section E.5 we modify some results of [Zanette and Brunskill, 2019] to our setting, and utilize the new results to bound each term in the regret decomposition in section E.6.

E.1 Properties of Confidence Intervals

In this section, we cite the important properties of the confidence intervals of EULER, as was stated in [Zanette and Brunskill, 2019]. We start from their definition of an admissible confidence interval:

Definition 1. *A confidence interval ϕ is called admissible for EULER if the following properties hold:*

1. *$\phi(p, V)$ takes the following functional form:*

$$\phi(p, V) = \frac{g(p, V)}{\sqrt{n_{k-1}(s, a) \vee 1}} + \frac{j(p, V)}{n_{k-1}(s, a) \vee 1} \quad ,$$

   for some functions $j(p, V) \leq J \in \mathbb{R}$, and

$$|g(p, V_1) - g(p, V_2)| \leq B_v \|V_1 - V_2\|_{2,p}$$

   If the value function is uniform then:

$$g(p, \alpha \mathbb{1}) = 0, \quad \forall \alpha \in \mathbb{R}.$$

2. *With probability at least $1 - \delta'$ it holds that:*

$$|(\hat{p}_{k-1}(\cdot \mid s, a) - p(\cdot \mid s, a))^T V^*_{t+1}| \leq \phi(p(\cdot \mid s, a), V^*_{t+1})$$

   jointly for all timesteps t, episodes k, states s and actions a.

3. *With probability at least $1 - \delta'$ it holds that:*

$$\left|g(\hat{p}_{k-1}(\cdot \mid s, a), V^*_{t+1}) - g(p(\cdot \mid s, a), V^*_{t+1})\right| \leq \frac{B_p}{\sqrt{n_{k-1}(s, a) \vee 1}}$$

   jointly for all episodes k, timesteps t, states s, actions a and some constant B_p that does not depend on $\sqrt{n_{k-1}(s, a) \vee 1}$.

An admissible confidence interval enjoys many properties, which are summarized in the following lemma:

Lemma 15. *If ϕ is admissible for EULER, and under the events that properties 2,3 of Definition 1 hold, then:*

1. *For any $V \in \mathbb{R}^S$ with $\|V\|_\infty \leq H$, it holds that $|g(p, V)| \leq B_v H$.*
2. *For any $V \in \mathbb{R}^S$,*

$$|\phi(\hat{p}_{k-1}(\cdot \mid s, a), V) - \phi(p(\cdot \mid s, a), V^*_{t+1})| \leq \frac{B_v \|V - V^*_{t+1}\|_{2,\hat{p}}}{\sqrt{n_{k-1}(s, a) \vee 1}} + \frac{B_p + 4J}{n_{k-1}(s, a) \vee 1}$$

3. *Let b^{pv}_k be the transition bonus, which is defined as*

$$b^{pv}_k(\hat{p}_{k-1}(\cdot \mid s, a), V_1, V_2) = \phi(\hat{p}_{k-1}(\cdot \mid s, a), V_1) + \frac{B_p + 4J}{n_{k-1}(s, a) \vee 1} + \frac{B_v \|V_2 - V_1\|_{2,\hat{p}}}{\sqrt{n_{k-1}(s, a) \vee 1}} \quad .$$

   For any $V_1, V_2 \in \mathbb{R}^S$ such that $V_1 \leq V^* \leq V_2$ pointwise, it holds that

$$b^{pv}_k(\hat{p}_{k-1}(\cdot \mid s, a), V_1, V_2) \geq \phi(p(\cdot \mid s, a), V^*)$$
$$b^{pv}_k(\hat{p}_{k-1}(\cdot \mid s, a), V_2, V_1) \geq \phi(p(\cdot \mid s, a), V^*)$$

Proof. The first property is due to Corollary 1.3 of [Zanette and Brunskill, 2019], and the second one is Lemma 4 of their paper. The third property is equivalent to Proposition 3 in [Zanette and Brunskill, 2019], but since we allow general value functions V_1, V_2, we write the full proof for completeness.

we start by proving that if $V_1 \leq V^* \leq V_2$, then for any transition probability vector p,

$$\begin{aligned} \|V_2 - V^*\|_{2,p} &\leq \|V_2 - V_1\|_{2,p} \\ \|V_1 - V^*\|_{2,p} &\leq \|V_2 - V_1\|_{2,p} \end{aligned} \tag{13}$$

To this end, notice that $\forall s$

$$0 \leq V_2(s) - V^*(s) \leq V_2(s) - V_1(s) \ ,$$

and since all of the quantities are non-negative, it also holds that

$$0 \leq \left(V_2(s) - V^*(s)\right)^2 \leq \left(V_2(s) - V_1(s)\right)^2 \ .$$

The inequality holds pointwise, and therefore holds for any linear combination with non-negative constants:

$$0 \leq \sum_s p(s)\left(V_2(s) - V^*(s)\right)^2 \leq \sum_s p(s)\left(V_2(s) - V_1(s)\right)^2 \ .$$

Taking the root of this inequality yields Inequality (13). Substituting in the definition of $b_k^{pv}(\hat{p}_{k-1}(\cdot \mid s,a), V_2, V_1)$ yields:

$$\begin{aligned} b_k^{pv}(\hat{p}_{k-1}(\cdot \mid s,a), V_2, V_1) &= \phi(\hat{p}_{k-1}(\cdot \mid s,a), V_2) + \frac{B_p + 4J}{n_{k-1}(s,a) \vee 1} + \frac{B_v \|V_2 - V_1\|_{2,\hat{p}}}{\sqrt{n_{k-1}(s,a) \vee 1}} \\ &\geq \phi(\hat{p}_{k-1}(\cdot \mid s,a), V_2) + \frac{B_p + 4J}{n_{k-1}(s,a) \vee 1} + \frac{B_v \|V_2 - V^*\|_{2,\hat{p}}}{\sqrt{n_{k-1}(s,a) \vee 1}} \\ &\geq \phi(p(\cdot \mid s,a), V^*) \end{aligned}$$

The first inequality is due to (13), and the second is due to the second part of the Lemma. The result for $b_k^{pv}(\hat{p}_{k-1}(\cdot \mid s,a), V_1, V_2)$ can be proven similarly, and thus omitted.

□

Another property that will be useful throughout the proof is the following upper bound on $b_k^{pv}(\hat{p}_{k-1}(\cdot \mid s,a), V_1, V_2)$

Lemma 16. *For any V_1, V_2 such that for all s, $V_1(s), V_2(s) \in [0, H]$*

$$b_k^{pv}(\hat{p}_{k-1}(\cdot \mid s,a), V_2, V_1) \leq \frac{2B_v H + 5J + B_p}{\sqrt{n_{k-1}(s,a) \vee 1}},$$

Proof. We bound $b_k^{pv}(\hat{p}_{k-1}(\cdot \mid s,a), V_2, V_1)$ as follows:

$$\begin{aligned} b_k^{pv}(\hat{p}_{k-1}(\cdot \mid s,a), V_2, V_1) &= \phi(\hat{p}_{k-1}(\cdot \mid s,a), V_2) + \frac{B_p + 4J}{n_{k-1}(s,a) \vee 1} + \frac{B_v \|V_2 - V_1\|_{2,\hat{p}}}{\sqrt{n_{k-1}(s,a) \vee 1}} \\ &\overset{(1)}{\leq} \frac{g(p,V)}{\sqrt{n_{k-1}(s,a) \vee 1}} + \frac{j(p,V)}{n_{k-1}(s,a) \vee 1} + \frac{B_p + 4J}{n_{k-1}(s,a) \vee 1} + \frac{B_v H}{\sqrt{n_{k-1}(s,a) \vee 1}} \\ &\overset{(2)}{\leq} \frac{B_v H}{\sqrt{n_{k-1}(s,a) \vee 1}} + \frac{J}{n_{k-1}(s,a) \vee 1} + \frac{B_p + 4J}{n_{k-1}(s,a) \vee 1} + \frac{B_v H}{\sqrt{n_{k-1}(s,a) \vee 1}} \\ &\overset{(3)}{\leq} \frac{2B_v H + 5J + B_p}{\sqrt{n_{k-1}(s,a) \vee 1}} \end{aligned}$$

In (1), we substituted ϕ and bounded $\|V_2 - V_1\|_{2,\hat{p}} \leq H$. (2) is by Lemma 15 and Definition 1, and (3) is by noting that $n \geq \sqrt{n}$ for $n \geq 1$. □

We end this section by stating that Bernstein's inequality induces an admissible ϕ. The proof can be found in [Zanette and Brunskill, 2019], Proposition 2.

Lemma 17. *Bernstein inequality induces an admissible confidence interval with* $g(p,V) = \sqrt{2\mathrm{Var}_{s'\sim p(\cdot|s,a)}V \ln\frac{2SAT}{\delta'}}$ *and* $j(p,V) = 2H\ln\frac{2SAT}{\delta'}$*, or explicitly:*

$$\phi(p(\cdot \mid s,a),V) = \sqrt{\frac{2\mathrm{Var}_{s'\sim p(\cdot|s,a)}V\ln\frac{2SAT}{\delta'}}{n_{k-1}(s,a)\vee 1}} + \frac{2H\ln\frac{2SAT}{\delta'}}{3n_{k-1}(s,a)\vee 1}$$

with the constants $J = \frac{2H\ln\frac{2SAT}{\delta'}}{3} = \tilde{\mathcal{O}}(H), B_v = \sqrt{2\ln\frac{2SAT}{\delta'}} = \tilde{\mathcal{O}}(1)$ *and* $B_p = H\sqrt{2\ln\frac{2SAT}{\delta'}} = \tilde{\mathcal{O}}(H)$*. Using lemma 16, it also implies that for any* V_1, V_2 *such that for all* s*,* $V_1(s), V_2(s) \in [0,H]$*, it holds that* $b_k^{pv}(\hat{p}_{k-1}(\cdot \mid s,a), V_2, V_1) \lesssim \tilde{\mathcal{O}}(H)$*.*

E.2 Failure Events

E.2.1 Failure Events of EULER

We start by recalling the failure events as stated in [Zanette and Brunskill, 2019], Appendix D. These events are high probability bounds that are based on the Empirical Bernstein Inequality [Maurer and Pontil, 2009] and leads to the bonus terms of the algorithm. Importantly, these events depend on the state-action visitation counter, and, thus, are indifferent to the greedy exploration scheme which we consider.

Define the following failure events.

$$F^r = \left\{ \exists s,a,k : \ |r(s,a) - \hat{r}_{k-1}(s,a)| \geq \sqrt{\frac{2\hat{\mathrm{Var}}_{k-1}R(s,a)\ln\frac{4SAT}{\delta'}}{n_{k-1}(s,a)\vee 1}} + \frac{14\ln\frac{4SAT}{\delta'}}{3(n_{k-1}(s,a)\vee 1)} \right\}$$

$$F^{vr} = \left\{ \exists s,a,k : \ \left| \sqrt{\hat{\mathrm{Var}}_{k-1}\ R(s,a)} - \sqrt{\mathrm{Var}\ R(s,a)} \right| \geq \sqrt{\frac{4\ln\frac{2SAT}{\delta'}}{n_{k-1}(s,a)\vee 1}} \right\}$$

$$F^{pv} = \left\{ \exists s,a,t,k : \ \left| (\hat{p}_{k-1}(\cdot \mid s,a) - p(\cdot \mid s,a))^T V_{t+1}^* \right| \geq \sqrt{\frac{2\mathrm{Var}_{s'\sim p(\cdot|s,a)}V_{t+1}^*\ln\frac{4SAT}{\delta'}}{n_{k-1}(s,a)\vee 1}} + \frac{2H\ln\frac{2SAT}{\delta'}}{3(n_{k-1}(s,a)\vee 1)} \right\}$$

$$F^{pv2} = \left\{ \exists s,a,t,k : \ |\|V_t^*\|_{2,\hat{p}} - \|V_t^*\|_{2,p}| \geq H\sqrt{\frac{4\ln\frac{2SAT}{\delta'}}{n_{k-1}(s,a)\vee 1}} \right\}$$

$$F^{ps} = \left\{ \exists s,s',a,k : \ |\hat{p}_{k-1}(s' \mid s,a) - p_{k-1}(s' \mid s,a)| \geq \sqrt{\frac{p(s' \mid s,a)(1-p(s' \mid s,a))\ln\frac{2TS^2A}{\delta'}}{n_{k-1}(s,a)\vee 1}} + \frac{2\ln\frac{2TS^2A}{\delta'}}{3(n_{k-1}(s,a)\vee 1)} \right\}$$

$$F^{pn1} = \left\{ \exists s,a,k : \ \|\hat{p}_{k-1}(\cdot \mid s,a) - p(\cdot \mid s,a)\|_1 \geq \sqrt{\frac{4S\ln\frac{3SAT}{\delta'}}{n_{k-1}(s,a)\vee 1}} \right\}$$

$$F_k^N = \left\{ \exists s,a,k : n_{k-1}(s,a) \leq \frac{1}{2}\sum_{j<k} w_j(s,a) - H\ln\frac{SAH}{\delta'} \right\}.$$

where $w_j(s,a) := \sum_{t=1}^H w_{tj}(s,a)$. In [Zanette and Brunskill, 2019], Appendix D, it is shown these events hold individually with probability at most δ'.

E.2.2 Failure Events of Decreasing Bounded Processes

In this section, we add another failure events to the total set of failure events. This set of failure event is not present in previous analysis of regret in optimistic RL algorithms (e.g., in Azar et al. 2017, Dann et al. 2017, 2018, Zanette and Brunskill 2019).

We define the following failure events.

$$F^{vDP} = \left\{ \exists K \geq 0 : \sum_{k=1}^{K} \sum_{t=1}^{H} \sum_{s} \bar{V}_t^{k-1}(s) - \mathbb{E}[\bar{V}_t^k(s) \mid \mathcal{F}_{k-1}] \geq 9SH^2 \ln \frac{3SH}{\delta'} \right\}$$

$$F^{vsDP} = \left\{ \exists K \geq 0 : \sum_{k=1}^{K} \sum_{t=1}^{H} \sum_{s} (\bar{V}_t^{k-1}(s) - \underline{V}_t^{k-1}(s))^2 - \mathbb{E}[(\bar{V}_t^k(s) - \underline{V}_t^k(s))^2 \mid \mathcal{F}_{k-1}] \geq 9SH^3 \ln \frac{3SH}{\delta'} \right\}$$

In this section, we prove that both of these failure events occur with low probability δ'.

We start by proving that $\left\{\bar{V}_t^k(s)\right\}$ is a decreasing processes, independently to the previously defined failure events. We continue and prove that $\left\{\bar{V}_t^k(s) - \underline{V}_t^k(s)\right\}^2$ starts as a decreasing process and then becomes and increasing process.

Lemma 18. *The following claims hold.*

1. *For every s, t, $\left\{\bar{V}_t^k(s)\right\}_k$ is a decreasing process and is bounded by $[0, H-(t-1)]$.*
2. *For every s, t, $\left\{\underline{V}_t^k(s)\right\}_k$ is an increasing process and is bounded by $[0, H-(t-1)]$.*
3. *For every s, t, $\left\{\left(\bar{V}_t^k(s) - \underline{V}_t^k(s)\right)^2\right\}_k$ starts as a decreasing process bounded by $[0, (H-(t-1)^2)]$ and then, possibly, becomes an increasing process.*

Proof. We start by proving the first claim.The following holds. By the initialization of the algorithm $\forall s, t,\ \bar{V}_t^0(s) = H-(t-1)$. By construction of the update rule $\bar{V}_t^k(s)$ can only decrease (see Line 14).

We now prove that for every s, t, k, $\left\{\bar{V}_t^k(s)\right\}_k$ is bounded from below by 0. By assumption $r(s,a) \in [0,1]$, and thus $\hat{r}_{k-1}(s,a) \geq 0$ a.s. . By induction, this implies $\bar{V}_t^{k-1} \geq 0$. The base case holds by initialization, and the induction step by the fact $\hat{r}_{k-1} \geq 0$ and that the bonus terms are positive.

Proving the second claim is done with similar argument, while using $\hat{r}_{k-1}(s,a) \leq 1$ a.s.. By the update rule (see Line 18), $\left\{\underline{V}_t^k(s)\right\}_k$ is an Increasing Bounded Process in $[0, H-(t-1)]$ (similar definition as in 1 with opposite inequality).

To prove the third claim we combine the two claims. Thus, $\left\{\left(\bar{V}_t^k(s) - \underline{V}_t^k(s)\right)^2\right\}_k$ starts as a decreasing process. Then, if the upper and lower value function crosses one another, the process becomes an increasing process. □

Remark 2. *Notice that the upper bound and lower bound of the optimal value crosses one another only inside the failure events defined in Section E.2.1). Yet, the analysis in the following will be indifferent to whether the failure event takes place or not.*

Lemma 19. $\Pr\{F^{vDP}\} \leq \delta'$.

Proof. We wish to bound

$$\Pr\{\exists K \geq 0 : \sum_{k=1}^{K} \sum_{t=1}^{H} \sum_{s} \bar{V}_h^{k-1}(s) - \mathbb{E}[\bar{V}_h^k(s) \mid \mathcal{F}_{k-1}] \geq 9SH^2 \ln \frac{3SH}{\delta'}\}.$$

According to Lemma 18, for every s, t, $\left\{\bar{V}_t^k(s)\right\}_{k \geq 1}$ is a decreasing process. Applying Lemma 11 (Appendix A) which bounds the sum of Decreasing Bounded Processes we conclude the proof. □

Lemma 20. $\Pr\{F^{vsDP}\} \leq \delta'$.

Proof. We wish to bound

$$\Pr\left\{\exists K \geq 0 : \sum_{k=1}^{K}\sum_{t=1}^{H}\sum_{s}(\bar{V}_t^{k-1}(s) - \underline{V}_t^{k-1}(s))^2 - \mathbb{E}[(\bar{V}_t^{k}(s) - \underline{V}_t^{k}(s))^2 \mid \mathcal{F}_{k-1}] \geq 9SH^3 \ln\frac{3SH}{\delta'}\right\}.$$

Consider a fixed s, t. Furthermore, define the following event

$$\mathbb{A}_{k-1} = \left\{\bar{V}_t^{k-1}(s) > \underline{V}_t^{k-1}(s)\right\}.$$

We have that

$$\begin{aligned}
&\sum_{k=1}^{K}(\bar{V}_t^{k-1}(s) - \underline{V}_t^{k-1}(s))^2 - \mathbb{E}[(\bar{V}_t^{k}(s) - \underline{V}_t^{k}(s))^2 \mid \mathcal{F}_{k-1}] \\
&\leq \sum_{k=1}^{K}\Big((\bar{V}_t^{k-1}(s) - \underline{V}_t^{k-1}(s))^2 - \mathbb{E}[(\bar{V}_t^{k}(s) - \underline{V}_t^{k}(s))^2 \mid \mathcal{F}_{k-1}]\Big)\mathbb{1}\{\mathbb{A}_{k-1}\} \\
&= \sum_{k=1}^{K}(\bar{V}_t^{k-1}(s) - \underline{V}_t^{k-1}(s))^2\mathbb{1}\{\mathbb{A}_{k-1}\} - \mathbb{E}[(\bar{V}_t^{k}(s) - \underline{V}_t^{k}(s))^2\mathbb{1}\{\mathbb{A}_{k-1}\} \mid \mathcal{F}_{k-1}] \\
&= \sum_{k=1}^{K}(\bar{V}_t^{k-1}(s) - \underline{V}_t^{k-1}(s))^2\mathbb{1}\{\mathbb{A}_{k-1}\} - \mathbb{E}[(\bar{V}_t^{k}(s) - \underline{V}_t^{k}(s))^2\mathbb{1}\{\mathbb{A}_{k}\} \mid \mathcal{F}_{k-1}] \\
&\quad - \sum_{k=1}^{K}\mathbb{E}[(\bar{V}_t^{k}(s) - \underline{V}_t^{k}(s))^2(\mathbb{1}\{\mathbb{A}_{k-1}\} - \mathbb{1}\{\mathbb{A}_{k}\}) \mid \mathcal{F}_{k-1}] \\
&\leq \sum_{k=1}^{K}(\bar{V}_t^{k-1}(s) - \underline{V}_t^{k-1}(s))^2\mathbb{1}\{\mathbb{A}_{k-1}\} - \mathbb{E}[(\bar{V}_t^{k}(s) - \underline{V}_t^{k}(s))^2\mathbb{1}\{\mathbb{A}_{k}\} \mid \mathcal{F}_{k-1}].
\end{aligned}$$

The first relation holds by definition, if the event $\mathbb{A}_{k-1}$ is false then the term is negative, since the process becomes increasing, and only decreases the sum. The second relation holds since $\mathbb{1}\{\mathbb{A}_{k-1}\}$ is $\mathcal{F}_{K-1}$ measureable. The forth relation holds since $(\bar{V}_t^{k-1}(s) - \underline{V}_t^{k-1}(s))^2 \geq 0$ and $(\mathbb{1}\{\mathbb{A}_{k-1}\} - \mathbb{1}\{\mathbb{A}_{k}\}) \geq 0$. Where the latter holds since $\mathbb{1}\{\mathbb{A}_{k}\} = 1 \rightarrow \mathbb{1}\{\mathbb{A}_{k-1}\} = 1$, i.e.,

$$\bar{V}_t^{k}(s) > \underline{V}_t^{k}(s) \;\rightarrow \bar{V}_t^{k-1}(s) > \underline{V}_t^{k-1}(s).$$

Differently put, if at the k^{th} episode $\bar{V}_t^{k}(s) > \underline{V}_t^{k}(s)$ then it also holds for the $k-1^{th}$ episode, $\bar{V}_t^{k-1}(s) > \underline{V}_t^{k}(s-1)$, as the process $\left\{\bar{V}_t^{k}(s)\right\}_{k\geq 0}$ is increasing and $\left\{\underline{V}_t^{k}(s)\right\}_{k\geq 0}$ is decreasing by Lemma 18.

Furthermore, by Lemma 18, $\left\{(\bar{V}_t^{k}(s) - \underline{V}_t^{k}(s))^2\mathbb{1}\{\mathbb{A}_{k}\}\right\}_k$ is a Decreasing Bounded Process in $[0, H^2]$. Initially, it decreases since $\mathbb{1}\{\mathbb{A}_{k}\}_k = 1$ and $\left\{(\bar{V}_t^{k}(s) - \underline{V}_t^{k}(s))^2\right\}$ is initially decreasing. Furthermore, when $\mathbb{1}\{\mathbb{A}_{k}\} = 0$ it cannot increase. Lastly, $(\bar{V}_t^{0}(s) - \underline{V}_t^{0}(s))^2\mathbb{1}\{\mathbb{A}_{0}\} \leq H^2$.

Applying Theorem 3 we get that for a fixed s, t, with probability $\frac{\delta'}{SH}$

$$\sum_{k=1}^{K}(\bar{V}_t^{k-1}(s) - \underline{V}_t^{k}(s))^2 - \mathbb{E}[(\bar{V}_t^{k}(s) - \underline{V}_t^{k-1}(s))^2 \mid \mathcal{F}_{k-1}] \geq 9SH^3 \ln\frac{3SH}{\delta'}.$$

By applying Lemma 11 (Appendix A), which extends this bound to the sum on s, t we conclude the proof. □

Lemma 21. *(All Failure Events) If $\delta' = \frac{\delta}{9}$, then*

$$F := F^{r}\bigcup F^{vr}\bigcup F^{pr}\bigcup F^{pv}\bigcup F^{pv2}\bigcup F^{ps}\bigcup F^{pn1}\bigcup F^{vDP}\bigcup F^{vsDP}$$

holds with probability at most δ. If the event F does not hold we say the algorithm is outside the failure event.

Proof. Applying a union bound on all events, which hold individually with probability at most δ' yield the result. □

E.3 EULER with Greedy Policies is Optimistic

Our algorithm modifies the exploration bonus of [Zanette and Brunskill, 2019] by using $\bar{V}_{k-1}, \underline{V}_{k-1}$ instead of $\bar{V}_k, \underline{V}_k$, and uses the following bonus (with some abuse of notation):

$$b_k^{pv}(s,a) = b_k^{pv}\left(\hat{p}_{k-1}(\cdot \mid s,a), \bar{V}_{k-1}, \underline{V}_{k-1}\right) \ .$$

We now show that the modified bonus retains the optimism of the algorithm:

Lemma 22. *Outside the failure event of the estimation (see Lemma 21), if the confidence interval is admissible, then the relation*

$$\underline{V}_t^{k-1} \leq V_t^* \leq \bar{V}_t^{k-1}$$

holds pointwise for all timesteps t and episodes k.

Proof. We follow the proof of [Zanette and Brunskill, 2019], Proposition 4, and prove by induction. We first prove that for all k, $V_t^* \leq \bar{V}_t^k$.

The claim trivially holds for $k = 0$, due to the initialization of the value. Suppose that the result holds for any state s and timestep t in the $k-1^{\text{th}}$ episode. If

$$\hat{r}_{k-1}(s_t^k, a_t^k) + b_{k-1}^r(s_t^k, a_t^k) + \hat{p}_{k-1}(\cdot \mid s, a_t^k)^T \bar{V}_{t+1}^{k-1} + b_{k-1}^{pv}(s_t^k, a_t^k) \geq \bar{V}_t^{k-1}(s_t) \ ,$$

then by the induction's assumption we are done. Otherwise, denote the optimal action in the real MDP at state s_t^k by a_t^*. The value is updated as follows:

$$\begin{aligned}
\bar{V}_k(s_t^k) &= \hat{r}_{k-1}(s_t^k, a_t^k) + b_{k-1}^r(s_t^k, a_t^k) + \hat{p}_{k-1}(\cdot \mid s, a_t^k)^T \bar{V}_{t+1}^{k-1} + b_{k-1}^{pv}(s_t^k, a_t^k) \\
&\geq \hat{r}_{k-1}(s_t^k a_t^*) + b_{k-1}^r(s_t^k, a_t^*) + \hat{p}_{k-1}(\cdot \mid s, a_t^*)^T \bar{V}_{t+1}^{k-1} + b_{k-1}^{pv}(s_t^k, a_t^*) \\
&\geq r(s_t^k, a_t^*) + \hat{p}_{k-1}(\cdot \mid s, a_t^*)^T \bar{V}_{t+1}^{k-1} + b_{k-1}^{pv}(s_t^k, a_t^*)
\end{aligned}$$

The first inequality is since a_t^k is the action that maximizes the greedy value and the second inequality is due to the optimism of the reward when the reward bonus is added, outside the failure events (Lemma 21). Next, using the inductive hypothesis ($V_{t+1}^* \leq \bar{V}_{t+1}^{k-1}$ element-wise), we get

$$\bar{V}_t^k(s_t^k) \geq r(s_t^k, a_t^*) + \hat{p}_{k-1}(\cdot \mid s, a_t^*)^T V_{t+1}^* + b_{k-1}^{pv}(s_t^k, a_t^*)$$

We now apply Lemma 15, which implies that

$$b_{k-1}^{pv}(s_t^k, a_t^*) \geq \phi(p(\cdot \mid s_t^k, a_t^*), V^*) \ ,$$

and thus

$$\bar{V}_t^k(s_t^k) \geq r(s_t^k, a_t^*) + \hat{p}_{k-1}(\cdot \mid s, a_t^*)^T V_{t+1}^* + \phi(p(\cdot \mid s_t^k, a_t^*), v^*)$$

Finally, since ϕ is admissible, we get the desired result from property (2) of Definition 1:

$$\bar{V}_t^k(s_t^k) \geq r(s_t^k, a_t^*) + p(\cdot \mid s, a_t^*)^T V_{t+1}^* = V_{t+1}^*(s_t^k)$$

The proof for $\underline{V}_t^{k-1} \leq V_t^*$ is almost identical, and thus omitted from this paper. □

E.4 Proof of Theorem 9

Proof. Throughout the proof, we assume that we are outside the failure events that were defined in Section E.2, which happens with probability of at least $1-\delta$ (Lemma 21). Specifically, it implies that the value function is optimistic, namely $V_1^*(s) \leq V_1^k(s)$ (Lemma 22), and we can bound the regret by,

$$\text{Regret}(K) = \sum_{k=1}^{K} V_1^*(s_1^k) - V_1^{\pi_k}(s_1^k) \leq \sum_{k=1}^{K} \bar{V}_1^{k-1}(s_1^k) - V_1^{\pi_k}(s_1^k).$$

Next, by applying Lemma 7 the following bound holds,

$$\leq \underbrace{\sum_{k=1}^{K}\sum_{t=1}^{H} \mathbb{E}[\bar{V}_t^{k-1}(s_t^k) - \bar{V}_t^k(s_t^k) \mid \mathcal{F}_{k-1}]}_{(A)}$$

$$+ \underbrace{\sum_{k=1}^{K}\sum_{t=1}^{H} \mathbb{E}\big[(\tilde{r}_{k-1} - r)(s_t^k, a_t^k) + (\tilde{p}_{k-1} - p)(\cdot \mid s_t^k, a_t^k)^T \bar{V}_{t+1}^{k-1} \mid \mathcal{F}_{k-1}\big]}_{(B)}. \tag{14}$$

The regret is thus upper bounded by two terms. The first term (A) also appears in the analysis of RTDP (Theorem 4). Specifically, by Lemma 34 (Appendix F), we can express this term as a sum of SH Decreasing Bounded Process in $[0, H]$:

$$(A) = \sum_{s}\sum_{t=1}^{H}\sum_{k=1}^{K} \bar{V}_t^{k-1}(s) - \mathbb{E}[\bar{V}_t^k(s) \mid \mathcal{F}_{k-1}].$$

Bounding (A). Outside failure event F^{vDP}, this term is bounded by $9SH^2 \ln \frac{3SH}{\delta'}$. Thus,

$$(A) \lesssim \tilde{\mathcal{O}}(SH^2)$$

Bounding (B). The term (B) is almost the same term that is bounded in [Zanette and Brunskill, 2019], and its presence is common in recent literature on exploration in RL (e.g., Dann et al. 2017, 2018, Zanette and Brunskill 2019). The only difference between (B) and the term bounded in [Zanette and Brunskill, 2019] is the presence of $\bar{V}^{k-1}$, the value *before* the update, instead of $\bar{V}^k$, the value after applying the update rule. This is since existing algorithms perform planning from the end of an episode and backwards. Thus, when choosing an action at some timestep t, these algorithms have access to the updated value of step $t+1$. In contrast, we avoid the planning stage, and therefore must rely on the previous value $\bar{V}^{k-1}$. We will later see that we can overcome this without affecting the regret.

Next, let L_k be the set of 'good' state-action pairs, which is defined in Definition 2 and analyzed thoroughly in Appendix F.1. We now decompose the sum of (B) to state actions in and outside L_k. We also note that except for the s_t^k, a_t^k, all of the variables in (B) are $\mathcal{F}_{k-1}$ measurable, which allows us to explicitly write the conditional expectation using $w_{tk}(s,a)$, as follows:

$$
\begin{aligned}
(B) &= \sum_{k=1}^{K}\sum_{t=1}^{H} w_{tk}(s,a)\Big((\tilde{r}_{k-1}-r)(s,a) + (\tilde{p}_{k-1}-p)(\cdot \mid s,a)^T \bar{V}_{t+1}^{k-1}\Big) \\
&= \sum_{k=1}^{K}\sum_{t=1}^{H}\sum_{(s,a)\in L_k} w_{tk}(s,a)\Big((\tilde{r}_{k-1}-r)(s,a) + (\tilde{p}_{k-1}-p)(\cdot \mid s,a)^T \bar{V}_{t+1}^{k-1}\Big) \\
&\quad + \sum_{k=1}^{K}\sum_{t=1}^{H}\sum_{(s,a)\notin L_k} w_{tk}(s,a)\Big((\tilde{r}_{k-1}-r)(s,a) + (\tilde{p}_{k-1}-p)(\cdot \mid s,a)^T \bar{V}_{t+1}^{k-1}\Big) \\
&\overset{(1)}{\lesssim} \sum_{k=1}^{K}\sum_{t=1}^{H}\sum_{(s,a)\in L_k} w_{tk}(s,a)\Big((\tilde{r}_{k-1}-r)(s,a) + (\tilde{p}_{k-1}-p)(\cdot \mid s,a)^T \bar{V}_{t+1}^{k-1}\Big) \\
&\quad + H\sum_{k=1}^{K}\sum_{t=1}^{H}\sum_{(s,a)\notin L_k} w_{tk}(s,a) \\
&\overset{(2)}{\lesssim} \sum_{k=1}^{K}\sum_{t=1}^{H}\sum_{(s,a)\in L_k} w_{tk}(s,a)\Big((\tilde{r}_{k-1}-r)(s,a) + (\tilde{p}_{k-1}-p)(\cdot \mid s,a)^T \bar{V}_{t+1}^{k-1}\Big) + \tilde{\mathcal{O}}(SAH^2)
\end{aligned}
$$

For (1), we bound $(\tilde{r}_{k-1}-r)(s,a) \le \tilde{r}_{k-1}(s,a)$ and $(\tilde{p}_{k-1}-p)(\cdot \mid s,a)^T \bar{V}_{t+1}^{k-1} \le \tilde{p}_{k-1}(\cdot \mid s,a)^T \bar{V}_{t+1}^{k-1}$. The estimated reward is in $[0,1]$ and it's bonus is at most $\tilde{\mathcal{O}}(1)$, and thus the optimistic reward $\tilde{r}_{k-1}(s,a)$ is $\tilde{\mathcal{O}}(1)$. Due to Lemma 22, the optimistic value $\bar{V}_{t+1}^{k-1} \le H$, and thus $\hat{p}_{k-1}(\cdot \mid s,a)^T \le H$. The transition bonus is $b_k^{pv}(s,a) = \tilde{\mathcal{O}}(H)$ due to Lemma 17, which implies that the second term is $\tilde{\mathcal{O}}(H)$. Together, both terms are $\tilde{\mathcal{O}}(H)$. (2) is due to Lemma 36 of Appendix F.1.

As in [Zanette and Brunskill, 2019], we continue the decomposition of the remaining term by adding and subtracting cross-terms that depends on $\hat{p}_{k-1}(\cdot \mid s,a)$

$$
\begin{aligned}
&\sum_{k=1}^{K}\sum_{t=1}^{H}\sum_{(s,a)\in L_k} w_{tk}(s,a)\big((\tilde{r}_{k-1}-r)(s,a) + (\tilde{p}_{k-1}-p)(\cdot \mid s,a)^T \bar{V}_{t+1}^{k-1}\big) \\
&= \sum_{k=1}^{K}\sum_{t=1}^{H}\sum_{(s,a)\in L_k} \underbrace{w_{tk}(s,a)(\tilde{r}_{k-1}-r)(s,a)}_{(1)} + \underbrace{w_{tk}(s,a)(\tilde{p}_{k-1}-\hat{p}_{k-1})^T(\cdot \mid s,a)\bar{V}_{t+1}^{k-1}}_{(2)} \\
&+ \underbrace{w_{tk}(s,a)(\hat{p}_{k-1}-p)(\cdot \mid s,a)^T V_{t+1}^{*}}_{(3)} + \underbrace{w_{tk}(s,a)(\hat{p}_{k-1}-p)(\cdot \mid s,a)^T(\bar{V}_{t+1}^{k-1}-V_{t+1}^{*})}_{(4)}. \qquad (15)
\end{aligned}
$$

Recall that we use Bernstein's inequality as the admissible confidence interval. Thus, by Lemma 26, it holds that $J = \frac{2H\ln\frac{2SAT}{\delta'}}{3} = \tilde{\mathcal{O}}(H), B_v = \sqrt{2\ln\frac{2SAT}{\delta'}} = \tilde{\mathcal{O}}(1)$ and $B_p = H\sqrt{2\ln\frac{2SAT}{\delta'}} = \tilde{\mathcal{O}}(H)$. Also let F, D be the constants defined in Lemma 23, and specifically

$$
\begin{aligned}
F &:= 2L + LH\sqrt{S} + 6B_vH = \tilde{\mathcal{O}}\Big(H\sqrt{S}\Big) \\
D &:= 18J + 4B_p + 4L^2 = \tilde{\mathcal{O}}(H)
\end{aligned}
$$

Substituting these constants, terms $(1)-(4)$ are bounded in Lemmas 30, 33, 31 and 32 respectively as follows:

$$
\begin{aligned}
(1) &\lesssim \sqrt{\mathbb{C}_r^* SAT} + SA \\
(2) &\lesssim \min\Big\{\sqrt{\mathbb{C}^* SAT} + S\sqrt{S}AH^2 + SAH^{\frac{5}{2}}, \sqrt{\mathbb{C}^\pi SAT} + S\sqrt{S}AH^{\frac{5}{2}}\Big\} \\
(3) &\lesssim \min\Big\{\sqrt{\mathbb{C}^* SAT} + SAH, \sqrt{\mathbb{C}^\pi SAT} + S\sqrt{S}AH^{\frac{5}{2}}\Big\} \\
(4) &\lesssim S^2AH^2 + S\sqrt{S}AH^{\frac{5}{2}}
\end{aligned}
$$

Thus, term (B) of the regret is bounded by:

$$\begin{aligned}(B) &\lesssim \min\Big\{\sqrt{\mathbb{C}^* SAT}, \sqrt{\mathbb{C}^\pi SAT}\Big\} + \sqrt{\mathbb{C}_r^* SAT} + S^2AH^2 + S\sqrt{S}AH^{\frac{5}{2}} \\ &\lesssim \sqrt{\min\{\mathbb{C}^* + \mathbb{C}_r^*, \mathbb{C}^\pi + \mathbb{C}_r^*\}SAT} + S\sqrt{S}AH^2\Big(\sqrt{S} + \sqrt{H}\Big)\end{aligned}$$

Finally, using Lemma 28, we can bound this term by

$$(B) \lesssim \sqrt{\min\Big\{\mathbb{Q}^*, \frac{\mathcal{G}^2}{H}\Big\}SAT} + S\sqrt{S}AH^2\Big(\sqrt{S} + \sqrt{H}\Big)$$

and noticing that (A) is negligible compared to (B), we get

$$\mathrm{Regret}(K) \lesssim \sqrt{\min\Big\{\mathbb{Q}^*, \frac{\mathcal{G}^2}{H}\Big\}SAT} + S\sqrt{S}AH^2\Big(\sqrt{S} + \sqrt{H}\Big)$$

To derive the problem independent bound, we use the fact that the maximal reward in a trajectory is bounded by $\mathcal{G} \leq H$, which yields

$$\mathrm{Regret}(K) \lesssim \sqrt{HSAT} + S\sqrt{S}AH^2\Big(\sqrt{S} + \sqrt{H}\Big)$$

□

E.5 Cumulative Squared Value Difference

In this section, we aim to bound the expected cumulative squared value difference. Specifically, we are interested in a bound for the following quantities:

$$\sum_{k=1}^{K}\sum_{t=1}^{H}\sum_{s,a} w_{tk}(s,a)p(\cdot \mid s,a)^T\Big(\bar{V}_{t+1}^{k-1} - \underline{V}_{t+1}^{k-1}\Big)^2. \tag{16}$$

$$\sum_{k=1}^{K}\sum_{t=1}^{H}\sum_{s,a} w_{tk}(s,a)p(\cdot \mid s,a)^T\big(\bar{V}_{t+1}^{k-1} - V_{t+1}^{\pi_k}\big)^2 \tag{17}$$

The first quantity allows us to replace Lemma 12 [Zanette and Brunskill, 2019], and the second allows us to prove Lemma 14 of the same paper. Together, they enable us to use the same analysis of [Zanette and Brunskill, 2019]. The final results are stated in Lemmas 26 and 27 by the end of this section. Most of the section will focus on bounding (16), which requires a much more delicate analysis than the bound of [Zanette and Brunskill, 2019].

In order to bound (16), we start by bounding $\Big(\bar{V}_{t+1}^{k-1} - \underline{V}_{t+1}^{k-1}\Big)^2$ in the following lemma, which corresponds to Proposition 5 of [Zanette and Brunskill, 2019]:

Lemma 23. *Outside the failure event, the following holds:*

$$\bar{V}_t^k(s_t^k) - \underline{V}_t^k(s_t^k) \leq \mathbb{E}[\bar{V}_{t+1}^{k-1}(s_{t+1}^k) - \underline{V}_{t+1}^{k-1}(s_{t+1}^k) \mid \mathcal{F}_{k-1}, s_t^k] + \min\left\{\frac{F+D}{\sqrt{n_{k-1}(s_t^k, a_t^k) \vee 1}}, H\right\},$$

where $F := 2L + LH\sqrt{S} + 6B_vH,\ D := 18J + 4B_p + 4L^2$, the constants J, B_v, B_p are defined in Definition 1 and $L := 2\sqrt{\ln \frac{4SAT}{\delta'}}$.

Proof. The proof is similar to [Zanette and Brunskill, 2019] Proposition 5, which is presented here with the needed adaptation.

If the state s_t^k is encountered in the k^{th} episode at the t^{th} time-step, then $\bar{V}_t^k(s_t^k), \underline{V}_t^k(s_t^k)$ will be updated according to the update rule. Thus,

$$\bar{V}_t^k(s_t^k) \leq \hat{r}_{k-1}(s_t^k, a_t^k) + b_{k-1}^r(s_t^k, a_t^k) + \hat{p}_{k-1}(\cdot \mid s_t^k, a_t^k)^T \bar{V}_{t+1}^{k-1} + b_k^{pv}(\hat{p}_{k-1}(\cdot \mid s_t^k, a_t^k), \bar{V}_{t+1}^{k-1}, \underline{V}_{t+1}^{k-1})$$
$$\underline{V}_t^k(s_t^k) \geq \hat{r}_{k-1}(s_t^k, a_t^k) - b_{k-1}^r(s_t^k, a_t^k) + \hat{p}_{k-1}(\cdot \mid s_t^k, a_t^k)^T \underline{V}_{t+1}^{k-1} - b_k^{pv}(\hat{p}_{k-1}(\cdot \mid s_t^k, a_t^k), \underline{V}_{t+1}^{k-1}, \bar{V}_{t+1}^{k-1}).$$

Subtraction yields:

$$\begin{aligned}\bar{V}_t^k(s_t^k) - \underline{V}_t^k(s_t^k) \leq\ & 2b_{k-1}^r(s_t^k, a_t^k) + \hat{p}_{k-1}(\cdot \mid s_t^k, a_t^k)^T(\bar{V}_{t+1}^{k-1} - \underline{V}_{t+1}^{k-1}) \\ & + b_k^{pv}(\hat{p}_{k-1}(\cdot \mid s_t^k, a_t^k), \bar{V}_{t+1}^{k-1}, \underline{V}_{t+1}^{k-1}) + b_k^{pv}(\hat{p}_{k-1}(\cdot \mid s_t^k, a_t^k), \underline{V}_{t+1}^{k-1}, \bar{V}_{t+1}^{k-1}).\end{aligned}$$

Next, we substitute the definition of the confidence bonus, which yields

$$\begin{aligned}\bar{V}_t^k(s_t^k) - \underline{V}_t^k(s_t^k) \leq & 2b_{k-1}^r(s_t^k, a_t^k) + \hat{p}_{k-1}(\cdot \mid s_t^k, a)^T(\bar{V}_{t+1}^{k-1} - \underline{V}_{t+1}^{k-1}) \\ & + \phi(\hat{p}_{k-1}(\cdot \mid s_t^k, a_t^k), \bar{V}_{t+1}^{k-1}) + \frac{4J + B_p}{n_{k-1}(s_t^k, a_t^k) \vee 1} + \frac{B_v \|\bar{V}_{t+1}^{k-1} - \underline{V}_{t+1}^{k-1}\|_{2,\hat{p}}}{\sqrt{n_{k-1}(s_t^k, a_t^k) \vee 1}} \\ & + \phi(\hat{p}_{k-1}(\cdot \mid s_t^k, a_t^k), \underline{V}_{t+1}^{k-1}) + \frac{4J + B_p}{n_{k-1}(s_t^k, a_t^k) \vee 1} + \frac{B_v \|\bar{V}_{t+1}^{k-1} - \underline{V}_{t+1}^{k-1}\|_{2,\hat{p}}}{\sqrt{n_{k-1}(s_t^k, a_t^k) \vee 1}}.\end{aligned}$$

Using Lemma 15, property (2), and Inequalities (13), we get,

$$\begin{aligned}\bar{V}_t^k(s_t^k) - \underline{V}_t^k(s_t^k) \leq & 2b_{k-1}^r(s_t^k, a_t^k) + \hat{p}_{k-1}(\cdot \mid s_t^k, a)^T(\bar{V}_{t+1}^{k-1} - \underline{V}_{t+1}^{k-1}) \\ & + 2\phi(p(\cdot \mid s_t^k, a_t^k), \underline{V}_{t+1}^*) + 4\left(\frac{4J + B_p}{n_{k-1}(s_t^k, a_t^k) \vee 1} + \frac{B_v \|\bar{V}_{t+1}^{k-1} - \underline{V}_{t+1}^{k-1}\|_{2,\hat{p}}}{\sqrt{n_{k-1}(s_t^k, a_t^k) \vee 1}}\right) \\ = & 2b_{k-1}^r(s_t^k, a_t^k) + p(\cdot \mid s_t^k, a)^T(\bar{V}_{t+1}^{k-1} - \underline{V}_{t+1}^{k-1}) \\ & + (\hat{p}_{k-1}(\cdot \mid s_t^k, a_t^k) - p(\cdot \mid s_t^k, a_t^k))^T(\bar{V}_{t+1}^{k-1} - \underline{V}_{t+1}^{k-1}) \\ & + 2\frac{g(p(\cdot \mid s_t^k, a_t^k), V_{t+1}^*)}{\sqrt{n_{k-1}(s_t^k, a_t^k) \vee 1}} + 2\frac{J}{n_{k-1}(s_t^k, a_t^k) \vee 1} \\ & + 4\left(\frac{4J + B_p}{n_{k-1}(s_t^k, a_t^k) \vee 1} + \frac{B_v \|\bar{V}_{t+1}^{k-1} - \underline{V}_{t+1}^{k-1}\|_{2,\hat{p}}}{\sqrt{n_{k-1}(s_t^k, a_t^k) \vee 1}}\right),\end{aligned}$$

where in the last relation we substituted ϕ and added and subtracted $p(\cdot \mid s_t^k, a)^T(\bar{V}_{t+1}^{k-1} - \underline{V}_{t+1}^{k-1})$.

By Lemma 18, we know that $\bar{V}_{t+1}^{k-1}, \underline{V}_{t+1}^{k-1} \in [0, H]$. Thus, $\left\|\bar{V}_{t+1}^{k-1} - \underline{V}_{t+1}^{k-1}\right\| \leq H$, which also implies that $\|\bar{V}_{t+1}^{k-1} - \underline{V}_{t+1}^{k-1}\|_{2,\hat{p}} \leq H$. In addition, using Hölder's inequality, and outside failure event F^{pn1}, we can bound

$$\begin{aligned}& (\hat{p}_{k-1}(\cdot \mid s_t^k, a_t^k) - p(\cdot \mid s_t^k, a_t^k))^T(\bar{V}_{t+1}^{k-1} - \underline{V}_{t+1}^{k-1}) \\ & \qquad \leq \left\|\hat{p}_{k-1}(\cdot \mid s_t^k, a_t^k) - p(\cdot \mid s_t^k, a_t^k)\right\|_1 \left\|\bar{V}_{t+1}^{k-1} - \underline{V}_{t+1}^{k-1}\right\|_\infty \\ & \qquad \leq H\sqrt{\frac{4S \ln \frac{2SAT}{\delta'}}{n_{k-1}(s, a) \vee 1}} = LH\sqrt{\frac{S}{n_{k-1}(s_t^k, a_t^k) \vee 1}}\end{aligned}$$

Substituting both of these bounds, we get

$$\bar{V}_t^k(s_t^k) - \underline{V}_t^k(s_t^k) \leq 2b_{k-1}^r(s_t^k, a_t^k) + p(\cdot \mid s_t^k, a)^T(\bar{V}_{t+1}^{k-1} - \underline{V}_{t+1}^{k-1}) + LH\sqrt{\frac{S}{n_{k-1}(s_t^k, a_t^k) \vee 1}}$$
$$+ 2\frac{g(p(\cdot \mid s_t^k, a_t^k), V_{t+1}^*)}{\sqrt{n_{k-1}(s_t^k, a_t^k) \vee 1}} + 2\frac{J}{n_{k-1}(s_t^k, a_t^k) \vee 1}$$
$$+ 4\left(\frac{4J + B_p}{n_{k-1}(s_t^k, a_t^k) \vee 1} + \frac{B_v H}{\sqrt{n_{k-1}(s_t^k, a_t^k) \vee 1}}\right) \tag{18}$$

We now bound the remaining terms. First, using Lemma 15, property (1), we can bound $g(p, V_{t+1}^*) \leq B_v H$. Second, notice that

$$p(\cdot \mid s_t^k, a))^T(\bar{V}_{t+1}^{k-1} - \underline{V}_{t+1}^{k-1}) = \sum_{s_{t+1}^k} p(s_{t+1}^k \mid s_t^k, a))^T(\bar{V}_{t+1}^{k-1}(s_{t+1}^k) - \underline{V}_{t+1}^{k-1}(s_{t+1}^k))$$
$$= \mathbb{E}\Big[\bar{V}_{t+1}^{k-1}(s_{t+1}^k) - \underline{V}_{t+1}^{k-1}(s_{t+1}^k) \mid \mathcal{F}_{k-1}, s_t^k\Big].$$

Finally, outside failure event F^r, the reward bonus can be bounded by

$$b_k^r(s_t^k, a_t^k) = \sqrt{\frac{2\hat{\mathrm{Var}}(R(s_t^k, a_t^k)) \ln \frac{4SAT}{\delta'}}{n_{k-1}(s_t^k, a_t^k) \vee 1}} + \frac{14 \ln \frac{4SAT}{\delta'}}{3n_{k-1}(s_t^k, a_t^k) \vee 1}$$
$$\leq \frac{L}{\sqrt{n_{k-1}(s_t^k, a_t^k) \vee 1}} + \frac{2L^2}{n_{k-1}(s_t^k, a_t^k) \vee 1}$$

where we used the fact that for variables in $[0, 1]$, $\hat{\mathrm{Var}}(R(s_t^k, a_t^k)) \leq 1$.

Putting it all together in (18), we get

$$\bar{V}_t^k(s_t^k) - \underline{V}_t^k(s_t^k) \leq \mathbb{E}\Big[\bar{V}_{t+1}^{k-1}(s_{t+1}^k) - \underline{V}_{t+1}^{k-1}(s_{t+1}^k) \mid \mathcal{F}_{k-1}, s_t^k\Big]$$
$$+ \frac{2L + LH\sqrt{S} + 6B_v H}{\sqrt{n_{k-1}(s_t^k, a_t^k) \vee 1}} + \frac{18J + 4B_p + 4L^2}{n_{k-1}(s_t^k, a_t^k) \vee 1}$$
$$\leq \mathbb{E}\Big[\bar{V}_{t+1}^{k-1}(s_{t+1}^k) - \underline{V}_{t+1}^{k-1}(s_{t+1}^k) \mid \mathcal{F}_{k-1}, s_t^k\Big] + \frac{F + D}{n_{k-1}(s_t^k, a_t^k) \vee 1}$$

where in the last relation we substituted F and D and used $\sqrt{n} \leq n$ for $n \geq 1$.

To finalize the proof note that outside the failure event, $\underline{V}_t^k(s) \leq \bar{V}_t^k(s)$, and the first term is therefore positive. combined with $\bar{V}_t^k(s_t^k) - \underline{V}_t^k(s_t^k) \leq H$ yields

$$\bar{V}_t^k(s_t^k) - \underline{V}_t^k(s_t^k) \leq \min\left\{\mathbb{E}\Big[\bar{V}_{t+1}^{k-1}(s_{t+1}^k) - \underline{V}_{t+1}^{k-1}(s_{t+1}^k) \mid \mathcal{F}_{k-1}, s_t^k\Big] + \frac{F + D}{\sqrt{n_{k-1}(s_t^k, a_t^k) \vee 1}}, H\right\}$$
$$\leq \mathbb{E}\Big[\bar{V}_{t+1}^{k-1}(s_{t+1}^k) - \underline{V}_{t+1}^{k-1}(s_{t+1}^k) \mid \mathcal{F}_{k-1}, s_t^k\Big] + \min\left\{\frac{F + D}{\sqrt{n_{k-1}(s_t^k, a_t^k) \vee 1}}, H\right\}.$$

□

Remark 3. *See that the first term in Equation Lemma 23 does not appear in the analysis of [Zanette and Brunskill, 2019]. Its existence is a direct consequence of the fact we use 1-step greedy policies, and not solving the approximate model at the beginning of each episode. Remarkably, we will later see that this term is comparable to the other previously existing terms.*

We now move to bounding the expected squared value difference, as formally stated in as follows:

Lemma 24. *Let* $\Delta_t^k := \Big(\bar{V}_t^{k-1}(s_t^k) - \underline{V}_t^{k-1}(s_t^k)\Big) - \Big(\bar{V}_t^{k}(s_t^k) - \underline{V}_t^{k}(s_t^k)\Big)$*. Then, outside the failure event,*

$$\begin{aligned}&\mathbb{E}[\Big(\bar{V}_t^{k-1}(s_t^k) - \underline{V}_t^{k-1}(s_t^k)\Big)^2 \mid \mathcal{F}_{k-1}] \\ &\leq 2H \sum_{t'=t}^{H-1} \mathbb{E}\Bigg[{\Delta_{t'}^k(s_{t'}^k)}^2 + \min\Bigg\{\frac{(F+D)^2}{n_{k-1}(s_{t'}^k, a_{t'}^k) \vee 1}, H^2\Bigg\} \mid \mathcal{F}_{k-1}\Bigg],\end{aligned}$$

where $F + D$ *is defined in Lemma 23.*

Proof. Before proving the bound, we express the bound of Lemma 23 in terms of Δ_t^k. For brevity, we denote $Y_k(s,a) := \min\Big\{\frac{F+D}{\sqrt{n_k(s,a)}\vee 1}, H\Big\}$, which is $\mathcal{F}_k$ measurable.

Assume the state s_t^k is visited in the k^{th} episode at the t^{th} time-step. Then, by Lemma 23,

$$\begin{aligned}\bar{V}_t^{k-1}(s_t^k) - \underline{V}_6^{k-1}(s_t^k) &= \Delta_t^k + \bar{V}_t^k(s_t^k) - \underline{V}_t^k(s_t^k) \\ &\leq \Delta_t^k + Y_{k-1}(s_t^k, a_t^k) + \underbrace{\mathbb{E}\Big[\bar{V}_{t+1}^{k-1}(s_{t+1}^k) - \underline{V}_{t+1}^{k-1}(s_{t+1}^k) \mid \mathcal{F}_{k-1}, s_t^k\Big]}_{(*)} . \end{aligned} \tag{19}$$

Next, by substituting Equation (19) in (*), we get

$$\begin{aligned}(*) &\leq \mathbb{E}\Big[\Delta_{t+1}^k + Y_{k-1}(s_{t+1}^k, a_{t+1}^k) + \mathbb{E}\Big[\bar{V}_{t+2}^{k-1}(s_{t+2}^k) - \underline{V}_{t+2}^{k-1}(s_{t+2}^k) \mid \mathcal{F}_{k-1}, s_{t+1}^k\Big] \mid \mathcal{F}_{k-1}, s_t^k\Big] \\ &= \mathbb{E}\Big[\Delta_{t+1}^k + Y_{k-1}(s_{t+1}^k, a_{t+1}^k) + \bar{V}_{t+2}^{k-1}(s_{t+2}^k) - \underline{V}_{t+2}^{k-1}(s_{t+2}^k) \mid \mathcal{F}_{k-1}, s_t^k\Big],\end{aligned}$$

where the last relation holds by the tower property.

Iterating using this technique until $t = H$, and using $\bar{V}_{H+1} = \underline{V}_{H+1} = 0$, we conclude the following bound:

$$\bar{V}_t^{k-1}(s_t^k) - \underline{V}_t^{k-1}(s_t^k) \leq \sum_{t'=t}^{H} \mathbb{E}[\Delta_{t'}^k(s_{t'}^k) + Y_{k-1}(s_{t'}^k, a_{t'}^k) \mid \mathcal{F}_{k-1}, s_t^k],$$

With this bound at hand, we can derive the desired result as follows:

$$\begin{aligned}\Big(\bar{V}_t^{k-1}(s_t^k) - \underline{V}_t^{k-1}(s_t^k)\Big)^2 &\leq \Bigg(\sum_{t'=t}^{H} \mathbb{E}\big[\Delta_{t'}^k(s_{t'}^k) + Y_{k-1}(s_{t'}^k, a_{t'}^k) \mid \mathcal{F}_{k-1}, s_t^k\big]\Bigg)^2 \\ &\overset{(CS)}{\leq} (H-t+1) \sum_{t'=t}^{H} \mathbb{E}\big[\Delta_{t'}^k(s_{t'}^k) + Y_{k-1}(s_{t'}^k, a_{t'}^k) \mid \mathcal{F}_{k-1}, s_t^k\big]^2 \\ &\overset{(J)}{\leq} (H-t+1) \sum_{t'=t}^{H} \mathbb{E}\Big[\big(\Delta_{t'}^k(s_{t'}^k) + Y_{k-1}(s_{t'}^k, a_{t'}^k)\big)^2 \mid \mathcal{F}_{k-1}, s_t^k\Big] \\ &\overset{(CS)}{\leq} 2(H-t+1) \sum_{t'=t}^{H} \mathbb{E}\big[\Delta_{t'}^k(s_{t'}^k)^2 + Y_{k-1}^2(s_{t'}^k, a_{t'}^k) \mid \mathcal{F}_{k-1}, s_t^k\big] \\ &\leq 2H \sum_{t'=t}^{H} \mathbb{E}\big[\Delta_{t'}^k(s_{t'}^k)^2 + Y_{k-1}^2(s_{t'}^k, a_{t'}^k) \mid \mathcal{F}_{k-1}, s_t^k\big]\end{aligned}$$

(CS) denotes Cauchy-Schwarz inequality, and specifically $\big(\sum_{i=1}^n a_i\big)^2 \leq n \sum_{i=1}^n a_i^2$. (J) is Jensen's inequality. Taking the conditional expectation $\mathbb{E}[\cdot \mid \mathcal{F}_{k-1}]$, using the tower property and substituting $Y_k(s,a)$ gives the desired result. □

After bounding the expected squared value difference in a single state, we now move to bounding its sum over different time-steps and episode. The main difficulty is in bounding the sum over the first term, which we bound in the following lemma:

Lemma 25. *Outside the failure event,*

$$\sum_{k=1}^{K}\sum_{t=1}^{H}\sum_{t'=t}^{H-1}\mathbb{E}[\Delta_{t'}^{k}(s_{t'}^{k})^{2}\mid\mathcal{F}_{k-1}]\leq\tilde{\mathcal{O}}(SH^{4}),$$

where $\Delta_t^k(s_t^k)$ is defined in Lemma 24.

Proof. We have that

$$\sum_{t=1}^{H}\sum_{t'=t}^{H}\mathbb{E}[\Delta_{t'}^{k}(s_{t'}^{k})^{2}\mid\mathcal{F}_{k-1}]=\sum_{t=1}^{H}t\mathbb{E}[\Delta_{t}^{k}(s_{t}^{k})^{2}\mid\mathcal{F}_{k-1}]\leq H\sum_{t=1}^{H}\mathbb{E}[\Delta_{t}^{k}(s_{t}^{k})^{2}\mid\mathcal{F}_{k-1}].$$

Furthermore,

$$\begin{aligned}
\left(\Delta_{t}^{k}(s_{t}^{k})\right)^{2} &= \left(\left(\bar{V}_{t}^{k-1}(s_{t}^{k})-\underline{V}_{t}^{k-1}(s_{t}^{k})\right)-\left(\bar{V}_{t}^{k}(s_{t}^{k})-\underline{V}_{t}^{k}(s_{t}^{k})\right)\right)^{2}\\
&= \left(\bar{V}_{t}^{k-1}(s_{t}^{k})-\underline{V}_{t}^{k-1}(s_{t}^{k})\right)^{2}+\left(\bar{V}_{t}^{k}(s_{t}^{k})-\underline{V}_{t}^{k}(s_{t}^{k})\right)^{2}\\
&\quad-2\left(\bar{V}_{t}^{k-1}(s_{t}^{k})-\underline{V}_{t}^{k-1}(s_{t}^{k})\right)\left(\bar{V}_{t}^{k}(s_{t}^{k})-\underline{V}_{t}^{k}(s_{t}^{k})\right)\\
&\leq \left(\bar{V}_{t}^{k-1}(s_{t}^{k})-\underline{V}_{t}^{k-1}(s_{t}^{k})\right)^{2}+\left(\bar{V}_{t}^{k}(s_{t}^{k})-\underline{V}_{t}^{k}(s_{t}^{k})\right)^{2}-2\left(\bar{V}_{t}^{k}(s_{t}^{k})-\underline{V}_{t}^{k}(s_{t}^{k})\right)^{2}\\
&= \left(\bar{V}_{t}^{k-1}(s_{t}^{k})-\underline{V}_{t}^{k-1}(s_{t}^{k})\right)^{2}-\left(\bar{V}_{t}^{k}(s_{t}^{k})-\underline{V}_{t}^{k}(s_{t}^{k})\right)^{2},
\end{aligned}$$

where the third relation holds since $\bar{V}^k(s), \underline{V}^k(s)$ decreases and increases, respectively, by Lemma 18, and since outside of the failure event $\bar{V}^k(s)\geq\underline{V}^k(s),\forall k$ (Lemma 22). Another implication these properties is that

$$\left(\bar{V}_{t}^{k-1}(s_{t}^{k})-\underline{V}_{t}^{k-1}(s_{t}^{k})\right)^{2}\geq\left(\bar{V}_{t}^{k}(s_{t}^{k})-\underline{V}_{t}^{k}(s_{t}^{k})\right)^{2},$$

Thus,

$$\begin{aligned}
&H\sum_{k=1}^{K}\sum_{t=1}^{H}\mathbb{E}[\Delta_{t}^{k}(s_{t}^{k})^{2}\mid\mathcal{F}_{k-1}]\\
&\leq H\sum_{k=1}^{K}\sum_{t=1}^{H}\mathbb{E}[\left(\bar{V}_{t}^{k-1}(s_{t}^{k})-\underline{V}_{t}^{k-1}(s_{t}^{k})\right)^{2}-\left(\bar{V}_{t}^{k}(s_{t}^{k})-\underline{V}_{t}^{k}(s_{t}^{k})\right)^{2}\mid\mathcal{F}_{k-1}]\ . \qquad (20)
\end{aligned}$$

For brevity, we define $\Delta V_t^k(s) = \bar{V}_t^k(s) - \underline{V}_t^k(s)$. Similarly to the technique used in Lemma 34 (Appendix F),

$$\begin{aligned}
&\sum_{k=1}^{K}\sum_{t=1}^{H}\mathbb{E}[\Big(\bar{V}_t^{k-1}(s_t^k) - \underline{V}_t^{k-1}(s_t^k)\Big)^2 - \Big(\bar{V}_t^{k}(s_t^k) - \underline{V}_t^{k}(s_t^k)\Big)^2 \mid \mathcal{F}_{k-1}] \\
&= \sum_{k=1}^{K}\sum_{t=1}^{H}\mathbb{E}[\Delta V_t^{k-1}(s_t^k)^2 - \Delta V_t^{k}(s_t^k)^2 \mid \mathcal{F}_{k-1}] \\
&\overset{(1)}{=} \sum_{k=1}^{K}\sum_{t=1}^{H}\sum_{s}\mathbb{E}[\mathbb{1}\{s_t^k = s\}\Delta V_t^{k-1}(s)^2 - \mathbb{1}\{s_t^k = s\}\Delta V_t^{k}(s)^2 \mid \mathcal{F}_{k-1}] \\
&\overset{(2)}{=} \sum_{k=1}^{K}\sum_{t=1}^{H}\sum_{s}\mathbb{E}[\mathbb{1}\{s_t^k = s\}\Delta V_t^{k-1}(s)^2 + \mathbb{1}\{s_t^k \neq s\}\Delta V_t^{k-1}(s)^2 \mid \mathcal{F}_{k-1}] \\
&\qquad\qquad - \mathbb{E}[\mathbb{1}\{s_t^k = s\}\Delta V_t^{k}(s)^2 + \mathbb{1}\{s_t^k \neq s\}\Delta V_t^{k-1}(s)^2 \mid \mathcal{F}_{k-1}] \\
&\overset{(3)}{=} \sum_{k=1}^{K}\sum_{t=1}^{H}\sum_{s}\Delta V_t^{k-1}(s)^2 - \mathbb{E}[\Delta V_t^{k}(s)^2 \mid \mathcal{F}_{k-1}]
\end{aligned}$$

Relation (1) holds by adding and subtracting $\mathbb{1}\{s \neq s_t^k\}\bar{V}_t^{k-1}(s)$ while using the linearity of expectation. (2) holds since for any event $\mathbb{1}\{A\} + \mathbb{1}\{A^c\} = 1$ and since ΔV_t^{k-1} is $\mathcal{F}_{k-1}$ measurable. (3) holds by the definition of the update rule. If state s is visited in the k^{th} episode at time-step t, then both $\bar{V}_t^k(s), \underline{V}_t^k(s)$ are updated. If not, their value remains as in the $k-1$ iteration.

Lastly,

$$\begin{aligned}
&\sum_{k=1}^{K}\sum_{t=1}^{H}\sum_{s}\Delta V_t^{k-1}(s)^2 - \mathbb{E}[\Delta V_t^{k}(s)^2 \mid \mathcal{F}_{k-1}] \\
&= \sum_{k=1}^{K}\sum_{t=1}^{H}\sum_{s}\Big(\bar{V}_t^{k-1}(s) - \underline{V}_t^{k-1}(s)\Big)^2 - \mathbb{E}[\Big(\bar{V}_t^{k}(s) - \underline{V}_t^{k}(s)\Big)^2 \mid \mathcal{F}_{k-1}] \leq \tilde{\mathcal{O}}(SH^3),
\end{aligned}$$

where the inequality holds outside the failure event F^{vsDP}, which is defined in Appendix E.2.2. Plugging this into (20) concludes the proof. □

We are now ready to prove the main results of this section and bound (16) and (17):

Lemma 26. *Outside the failure event.*

$$\sum_{k=1}^{K}\sum_{t=1}^{H}\sum_{s,a} w_{tk}(s,a)p(\cdot \mid s,a)^T\left(\bar{V}_{t+1}^{k-1}-\underline{V}_{t+1}^{k-1}\right)^2 \leq \tilde{\mathcal{O}}(SAH^2(F+D)^2+SAH^5).$$

where $F+D$ is defined in Lemma 23

Proof. Recall that $w_{tk}(s,a)=\Pr(s_t^k \mid s_1^k,\pi_k)$ is the probability when following π^k in the true MDP the state-action in the k^{th} episode at the t^{th} time-step is $(s_t^k,a_t^k)=(s,a)$. Thus, the following relation holds.

$$\begin{aligned}
&\sum_{s,a} w_{tk}(s,a)p(\cdot \mid s,a)^T(\bar{V}_{t+1}^{k-1}-\underline{V}_{t+1}^{k-1})^2\\
&=\sum_{s_t}\Pr(s_t^k \mid s_1^k,\pi_k)\sum_{s_{t+1}} p(s_{t+1}^k \mid s_t^k,a_t^k)(\bar{V}_{t+1}^{k-1}(s_{t+1}^k)-\underline{V}_{t+1}^{k-1}(s_{t+1}^k))^2\\
&=\sum_{s_{t+1}}\Pr(s_{t+1}^k \mid s_1^k,\pi_k)(\bar{V}_{t+1}^{k-1}(s_{t+1}^k)-\underline{V}_{t+1}^{k-1}(s_{t+1}^k))^2\\
&=\mathbb{E}[(\bar{V}_{t+1}^{k-1}(s_{t+1})-\underline{V}_{t+1}^{k-1}(s_{t+1}))^2 \mid \mathcal{F}_{k-1}].
\end{aligned}$$

Since $\bar{V}_{H+1}^{k-1}(s_{t+1})=\underline{V}_{H+1}^{k-1}(s_{t+1}=0$, we obtain,

$$\begin{aligned}
&\sum_{k=1}^{K}\sum_{t=1}^{H}\sum_{s,a} w_{tk}(s,a)p(\cdot \mid s,a)(\bar{V}_{t+1}^{k-1}-\underline{V}_{t+1}^{k-1})^2\\
&=\sum_{k=1}^{K}\sum_{t=1}^{H}\mathbb{E}[(\bar{V}_{t+1}^{k-1}(s_t^k)-\underline{V}_{t+1}^{k-1}(s_t^k)^2 \mid \mathcal{F}_{k-1}]\\
&\leq\sum_{k=1}^{K}\sum_{t=1}^{H}\mathbb{E}[(\bar{V}_{t}^{k-1}(s_t^k)-\underline{V}_{t}^{k-1}(s_t^k)^2 \mid \mathcal{F}_{k-1}].
\end{aligned}$$

Thus,

$$\begin{aligned}
&\sum_{k=1}^{K}\sum_{t=1}^{H}\sum_{s,a} w_{tk}(s,a)p(\cdot \mid s,a)^T\left(\bar{V}_{t+1}^{k-1}-\underline{V}_{t+1}^{k-1}\right)^2\\
&\leq\sum_{k=1}^{K}\sum_{t=1}^{H}\mathbb{E}[\left(\bar{V}_{t+1}^{k-1}(s_t^k)-\underline{V}_{t+1}^{k-1}(s_t^k)\right)^2 \mid \mathcal{F}_{k-1}]\\
&\overset{(*)}{\leq} 2H\sum_{k=1}^{K}\sum_{t=1}^{H}\sum_{t'=t}^{H}\mathbb{E}[\Delta_{t'}^k(s_{t'}^k)^2 \mid \mathcal{F}_{k-1}]+2H\sum_{k=1}^{K}\sum_{t=1}^{H}\sum_{t'=t}^{H}\mathbb{E}\left[\min\left\{\frac{(F+D)^2}{n_{k-1}(s_{t'}^k,a_{t'}^k)\vee 1},H^2\right\} \mid \mathcal{F}_{k-1}\right]\\
&=2H\sum_{k=1}^{K}\sum_{t=1}^{H}t\mathbb{E}[\Delta_t^k(s_t^k)^2 \mid \mathcal{F}_{k-1}]+2H\sum_{k=1}^{K}\sum_{t=1}^{H}t\mathbb{E}\left[\min\left\{\frac{(F+D)^2}{n_{k-1}(s_t^k,a_t^k)\vee 1},H^2\right\} \mid \mathcal{F}_{k-1}\right]\\
&\leq 2H^2\sum_{k=1}^{K}\sum_{t=1}^{H}\mathbb{E}[\Delta_t^k(s_t^k)^2 \mid \mathcal{F}_{k-1}]+2H^2\sum_{k=1}^{K}\sum_{t=1}^{H}\mathbb{E}\left[\min\left\{\frac{(F+D)^2}{n_{k-1}(s_t^k,a_t^k)\vee 1},H^2\right\} \mid \mathcal{F}_{k-1}\right] \qquad (21)
\end{aligned}$$

where $(*)$ last relation holds by Lemma 24, in which $\Delta_t^k(s_t^k)$ is defined. The first term is bounded in Lemma 25 by $\tilde{\mathcal{O}}(SH^5)$. The second term is bounded outside the failure event Using the 'Good Set' L_k, which is defined and analyzed in Appendix F.1. The bound for this term can be found in Lemma 39. Combining both of the results and substituting into (21) yields

$$\begin{aligned}
\sum_{k=1}^{K}\sum_{t=1}^{H}\sum_{s,a} w_{tk}(s,a)p(\cdot \mid s,a)^T\left(\bar{V}_{t+1}^{k-1}-\underline{V}_{t+1}^{k-1}\right)^2 &\leq \tilde{\mathcal{O}}(SH^5)+\tilde{\mathcal{O}}(SAH^2(F+D)^2+SAH^5)\\
&=\tilde{\mathcal{O}}(SAH^2(F+D)^2+SAH^5)
\end{aligned}$$

□

Lemma 27. *Outside the failure event.*

$$\sum_{k=1}^{K}\sum_{t=1}^{H}\sum_{s,a} w_{tk}(s,a)p(\cdot \mid s,a)^T\left(\bar{V}_{t+1}^{k-1} - V_{t+1}^{\pi_k}\right)^2 \leq \tilde{\mathcal{O}}(SAH^3(F+D)^2 + SAH^5)$$

where $F + D$ is defined in Lemma 23

Proof. Similarly to Lemma 26, we have that

$$\begin{aligned}
&\sum_{k=1}^{K}\sum_{t=1}^{H}\sum_{s,a} w_{tk}(s,a)p(\cdot \mid s,a)^T\left(\bar{V}_{t+1}^{k-1} - V_{t+1}^{\pi_k}\right)^2 \\
&\leq \sum_{k=1}^{K}\sum_{t=1}^{H}\mathbb{E}\Big[\left(\bar{V}_{t+1}^{k-1}(s_t^k) - V_{t+1}^{\pi_k}(s_t^k)\right)^2 \mid \mathcal{F}_{k-1}\Big] \\
&\leq \sum_{k=1}^{K}\sum_{t=1}^{H}\mathbb{E}\Big[\left(\bar{V}_{t}^{k-1}(s_t^k) - V_{t}^{\pi_k}(s_t^k)\right)^2 \mid \mathcal{F}_{k-1}\Big]. \qquad (22)
\end{aligned}$$

where the last inequality is since $\bar{V}_{H+1}^{k-1}(s_{t+1}) = V_{H+1}^{\pi_k}(s_{t+1} = 0$. Applying Lemma 7, we get,

$$\begin{aligned}
&\mathbb{E}[\left(\bar{V}_t^{k-1}(s_t^k) - V_t^{\pi_k}(s_t^k)\right)^2 \mid \mathcal{F}_{k-1}] \\
&\overset{(1)}{\leq} \mathbb{E}\left[\left(\sum_{t'=t}^{H}\mathbb{E}\Big[\bar{V}^{k-1}(s_{t'}^k) - \bar{V}^k(s_{t'}^k) + (\tilde{r}_{k-1} - r)(s_{t'}^k, a_{t'}^k) + (\tilde{p}_{k-1} - p)(s_{t'}^k, a_{t'}^k)\bar{V}_{t+1}^{k-1} \mid \mathcal{F}_{k-1}, s_t^k\Big]\right)^2 \mid \mathcal{F}_{k-1}\right] \\
&\overset{(2)}{\leq} 3H\mathbb{E}\left[\sum_{t'=t}^{H}\mathbb{E}\left[\left(\bar{V}^{k-1}(s_{t'}^k) - \bar{V}^k(s_{t'}^k)\right)^2 \mid \mathcal{F}_{k-1}, s_t^k\right] \mid \mathcal{F}_{k-1}\right] \\
&\quad + 3H\mathbb{E}\left[\sum_{t'=t}^{H}\mathbb{E}\left[\left((\tilde{r}_{k-1} - r)(s_{t'}^k, a_{t'}^k)\right)^2 + \left((\tilde{p}_{k-1} - p)(s_{t'}^k, a_{t'}^k)\bar{V}_{t+1}^{k-1}\right)^2 \mid \mathcal{F}_{k-1}, s_t^k\right] \mid \mathcal{F}_{k-1}\right] \\
&\overset{(3)}{=} 3H\sum_{t'=t}^{H}\mathbb{E}\left[\left(\bar{V}^{k-1}(s_{t'}^k) - \bar{V}^k(s_{t'}^k)\right)^2 \mid \mathcal{F}_{k-1}\right] \\
&\quad + 3H\sum_{t'=t}^{H}\mathbb{E}\left[\left((\tilde{r}_{k-1} - r)(s_{t'}^k, a_{t'}^k)\right)^2 + \left((\tilde{p}_{k-1} - p)(s_{t'}^k, a_{t'}^k)\bar{V}_{t+1}^{k-1}\right)^2 \mid \mathcal{F}_{k-1}\right].
\end{aligned}$$

Inequality (1) is by Lemma 12. (2) is due to Jensen's inequality, and using the inequality $\left(\sum_{i=1}^n a_i\right)^2 \leq n\sum_{i=1}^n a_i^2$, and (3) is by the tower property.
Plugging this back into (22),

$$\begin{aligned}
(22) &\leq 3H\sum_{k=1}^{K}\sum_{t=1}^{H}\sum_{t'=t}^{H}\mathbb{E}\Big[\left(\bar{V}^{k-1}(s_{t'}^k) - \bar{V}^k(s_{t'}^k)\right)^2 \mid \mathcal{F}_{k-1}\Big] \\
&\quad + 3H\sum_{k=1}^{K}\sum_{t=1}^{H}\sum_{t'=t}^{H}\mathbb{E}\Big[\left((\tilde{r}_{k-1} - r)(s_{t'}^k, a_{t'}^k)\right)^2 + \left((\tilde{p}_{k-1} - p)(s_{t'}^k, a_{t'}^k)\bar{V}_{t+1}^{k-1}\right)^2 \mid \mathcal{F}_{k-1}\Big] \\
&\leq 3H^2\sum_{k=1}^{K}\sum_{t=1}^{H}\underbrace{\mathbb{E}\Big[\left(\bar{V}^{k-1}(s_t^k) - \bar{V}^k(s_t^k)\right)^2 \mid \mathcal{F}_{k-1}\Big]}_{(*)} \\
&\quad + 3H^2\sum_{k=1}^{K}\sum_{t=1}^{H}\mathbb{E}\left[\underbrace{\left((\tilde{r}_{k-1} - r)(s_t^k, a_t^k)\right)^2}_{(**)} + \underbrace{\left((\tilde{p}_{k-1} - p)(s_t^k, a_t^k)\bar{V}_{t+1}^{k-1}\right)^2}_{(***)} \mid \mathcal{F}_{k-1}\right]. \qquad (23)
\end{aligned}$$

We now bound each term of the above. First, we have that

$$
\begin{aligned}
(*) &= \sum_{t=1}^{H}\sum_{t=1}^{H}\mathbb{E}\Big[\big(\bar{V}^{k-1}(s_t^k)-\bar{V}^{k}(s_t^k)\big)^2 \mid \mathcal{F}_{k-1}\Big] \\
&= \sum_{t=1}^{H}\sum_{t=1}^{H}\mathbb{E}\Big[\big(\bar{V}^{k-1}(s_t^k)\big)^2+\big(\bar{V}^{k}(s_t^k)\big)^2-2\bar{V}^{k}(s_t^k)\bar{V}^{k-1}(s_t^k) \mid \mathcal{F}_{k-1}\Big] \\
&\overset{(1)}{\leq} \sum_{t=1}^{H}\sum_{t=1}^{H}\mathbb{E}\Big[\big(\bar{V}^{k-1}(s_t^k)\big)^2+\big(\bar{V}^{k}(s_t^k)\big)^2-2\big(\bar{V}^{k}(s_t^k)\big)^2 \mid \mathcal{F}_{k-1}\Big] \\
&= \sum_{t=1}^{H}\sum_{t=1}^{H}\mathbb{E}\Big[\big(\bar{V}^{k-1}(s_t^k)\big)^2-\big(\bar{V}^{k}(s_t^k)\big)^2 \mid \mathcal{F}_{k-1}\Big] \\
&\overset{(2)}{=} \sum_{t=1}^{H}\sum_{t=1}^{H}\sum_{s}\big(\bar{V}^{k-1}(s)\big)^2-\mathbb{E}\Big[\big(\bar{V}^{k}(s)\big)^2 \mid \mathcal{F}_{k-1}\Big].
\end{aligned}
$$

Relation (1) holds since $0 \leq \bar{V}^k \leq \bar{V}^{k-1}$ (see Lemma 18). (2) is proven similarly to Lemma 34 (Appendix F), as follows

$$
\begin{aligned}
&\sum_{k=1}^{K}\sum_{t=1}^{H}\mathbb{E}\Big[\big(\bar{V}^{k-1}(s_t^k)\big)^2-\big(\bar{V}^{k}(s_t^k)\big)^2 \mid \mathcal{F}_{k-1}\Big] \\
&\overset{(1)}{=} \sum_{k=1}^{K}\sum_{t=1}^{H}\sum_{s}\mathbb{E}[\mathbb{1}\{s_t^k=s\}\big(\bar{V}^{k-1}(s)\big)^2-\mathbb{1}\{s_t^k=s\}\big(\bar{V}^{k}(s)\big)^2 \mid \mathcal{F}_{k-1}] \\
&\overset{(2)}{=} \sum_{k=1}^{K}\sum_{t=1}^{H}\sum_{s}\mathbb{E}[\mathbb{1}\{s_t^k=s\}\big(\bar{V}^{k-1}(s)\big)^2+\mathbb{1}\{s_t^k\neq s\}\big(\bar{V}^{k-1}(s)\big)^2 \mid \mathcal{F}_{k-1}] \\
&\qquad\qquad -\mathbb{E}[\mathbb{1}\{s_t^k=s\}\big(\bar{V}^{k}(s)\big)^2+\mathbb{1}\{s_t^k\neq s\}\big(\bar{V}^{k-1}(s)\big)^2 \mid \mathcal{F}_{k-1}] \\
&\overset{(3)}{=} \sum_{k=1}^{K}\sum_{t=1}^{H}\sum_{s}\big(\bar{V}^{k-1}(s)\big)^2-\mathbb{E}[\big(\bar{V}^{k}(s)\big)^2 \mid \mathcal{F}_{k-1}]
\end{aligned}
$$

(1) holds by adding and subtracting $\mathbb{1}\{s\neq s_t^k\}\bar{V}_t^{k-1}(s)$ while using the linearity of expectation. (2) holds since for any event $\mathbb{1}\{A\}+\mathbb{1}\{A^c\}=1$ and since ΔV_t^{k-1} is $\mathcal{F}_{k-1}$ measurable. (3) holds by the definition of the update rule. If state s is visited in the k^{th} episode at time-step t, then both $\bar{V}_t^k(s), \underline{V}_t^k(s)$ are updated. If not, their value remains as in the $k-1$ iteration.

Next, by Lemma 18 for a fixed s,t, $\big\{\bar{V}_t^k(s)\big\}_{k\geq 0}$ is a Decreasing Bounded Process in $[0,H^2]$. Applying Lemma 11 we conclude that

$$
(*) \leq \sum_{k=1}^{K}\sum_{t=1}^{H}\sum_{s}\big(\bar{V}^{k-1}(s)\big)^2-\mathbb{E}[\big(\bar{V}^{k}(s)\big)^2 \mid \mathcal{F}_{k-1}] \lesssim \tilde{\mathcal{O}}(H^3S).
$$

We now turn to bound $(**)$.

$$\begin{aligned}
&\sum_{k=1}^{K}\sum_{t=1}^{H}\mathbb{E}\Big[\big((\tilde{r}_{k-1}-r)(s_t^k,a_t^k)\big)^2 \mid \mathcal{F}_{k-1}\Big] \\
&\overset{(1)}{\leq} 2\sum_{k=1}^{K}\sum_{t=1}^{H}\mathbb{E}\Big[\big((\hat{r}_{k-1}-r)(s_t^k,a_t^k)\big)^2 \mid \mathcal{F}_{k-1}\Big] + 2\sum_{k=1}^{K}\sum_{t=1}^{H}\mathbb{E}\Big[\big(b_k^r(s_t^k,a_t^k)\big)^2 \mid \mathcal{F}_{k-1}\Big] \\
&\overset{(2)}{\leq} 4\sum_{k=1}^{K}\sum_{t=1}^{H}\mathbb{E}\Big[\big(b_k^r(s_t^k,a_t^k)\big)^2 \mid \mathcal{F}_{k-1}\Big] \\
&= 4\sum_{k=1}^{K}\sum_{t=1}^{H}\mathbb{E}\left[\left(\sqrt{\frac{2\hat{\mathrm{Var}}(R(s_t^k,a_t^k))\ln\frac{4SAT}{\delta'}}{n_{k-1}(s_t^k,a_t^k)\vee 1}}+\frac{14\ln\frac{4SAT}{\delta'}}{3n_{k-1}(s_t^k,a_t^k)\vee 1}\right)^2 \mid \mathcal{F}_{k-1}\right] \\
&\overset{(3)}{\lesssim} \sum_{k=1}^{K}\sum_{t=1}^{H}\mathbb{E}\left[\frac{1}{n_{k-1}(s_t^k,a_t^k)\vee 1} \mid \mathcal{F}_{k-1}\right] \\
&\overset{(4)}{\lesssim} \tilde{\mathcal{O}}(SAH).
\end{aligned}$$

In (1), we used the definition of $\tilde{r}_{k-1}$ and the inequality $(a+b)^2 \leq 2a^2+2b^2$. (2) is since outside the failure event F^r, $(\hat{r}_{k-1}-r)(s,a) \leq b_k^r(s,a)$. (3) uses the fact that $R(s,a)\in[0,1]$, and thus $\hat{\mathrm{Var}}(R(s,a)\leq 1$, and $\sqrt{n}\leq n$ for $n\geq 1$. Finally, (4) is due to Lemma 39.

Lastly, we bound $(***)$.

$$\begin{aligned}
&\sum_{k=1}^{K}\sum_{t=1}^{H}\mathbb{E}\Big[\big((\tilde{p}_{k-1}-p)(s_t^k,a_t^k)^T\bar{V}_{t+1}^{k-1}\big)^2 \mid \mathcal{F}_{k-1}\Big] \\
&= \sum_{k=1}^{K}\sum_{t=1}^{H}\mathbb{E}\Big[\big((\hat{p}_{k-1}-p)(s_t^k,a_t^k)^T\bar{V}_{t+1}^{k-1}+b_k^{pv}(s_t^k,a_t^k)\big)^2 \mid \mathcal{F}_{k-1}\Big] \\
&\overset{(1)}{\leq} 2\sum_{k=1}^{K}\sum_{t=1}^{H}\mathbb{E}\Big[\big((\hat{p}_{k-1}-p)(s_t^k,a_t^k)^T\bar{V}_{t+1}^{k-1}\big)^2+\big(b_k^{pv}(s_t^k,a_t^k)\big)^2 \mid \mathcal{F}_{k-1}\Big] \\
&\overset{(2)}{\leq} 2\sum_{k=1}^{K}\sum_{t=1}^{H}\mathbb{E}\Big[\big(\|\hat{p}_{k-1}-p\|_1\|\bar{V}_{t+1}^{k-1}\|_\infty\big)^2+\big(b_k^{pv}(s_t^k,a_t^k)\big)^2 \mid \mathcal{F}_{k-1}\Big] \\
&\overset{(3)}{\leq} 2\sum_{k=1}^{K}\sum_{t=1}^{H}\mathbb{E}\Big[H^2\|\hat{p}_{k-1}-p\|_1^2+\big(b_k^{pv}(s_t^k,a_t^k)\big)^2 \mid \mathcal{F}_{k-1}\Big] \\
&\overset{(4)}{\lesssim} \sum_{k=1}^{K}\sum_{t=1}^{H}\mathbb{E}\left[\frac{H^2S}{n_{k-1}(s_t^k,a_t^k)}+\frac{(2B_vH+5J+B_p)^2}{n_{k-1}(s_t^k,a_t^k)} \mid \mathcal{F}_{k-1}\right]. \\
&\overset{(5)}{\lesssim} \tilde{\mathcal{O}}(SAH(F+D)^2)
\end{aligned}$$

Similarly to the bound on the reward, (1) uses the inequality $(a+b)^2\leq 2a^2+2b^2$. Inequality (2) is due to Hölder's inequality, and (3) bounds $\|\bar{V}_{t+1}^{k-1}\|_\infty \leq H$, which is due to Lemma 18. Next, (4) bounds the transition error outside to failure event F^{pn1} and b_k^{pv} according to Lemma 16. Finally, (5) is by Lemma 39 and noting that $H^2S+(2B_vH+5J+B_p)^2 \lesssim (F+D)^2$.

Substituting all of the results into (23), and remembering the H^2 factor in this equation, gives the desired result.

□

E.6 Bounding Different Terms in the Regret Decomposition

In this section, we bound each of the individual terms of the regret decomposition (Equation 15), relaying on results from [Zanette and Brunskill, 2019], as well as on the new lemmas derived in Section E.5, Lemma 26 and Lemma 27. First, we present the problem dependent constants of [Zanette and Brunskill, 2019] for general admissible confidence intervals, and their relation to problem dependent constants with Bernstein's inequality

Lemma 28. *Let $\mathbb{C}^*$ and $\mathbb{C}^\pi$ be upper dependent bounds on the following qualities:*

$$\mathbb{C}^* \geq \frac{1}{T}\sum_{k=1}^{K}\sum_{t=1}^{H}\sum_{s,a} w_{tk}(s,a) g(p, V_{t+1}^*)^2$$
$$\mathbb{C}^\pi \geq \frac{1}{T}\sum_{k=1}^{K}\sum_{t=1}^{H}\sum_{s,a} w_{tk}(s,a) g(p, V_{t+1}^{\pi_k})^2 \quad ,$$

with $g(p,V) = \sqrt{2\mathrm{Var}_{s'\sim p(\cdot|s,a)}V(s')\ln\frac{2SAT}{\delta'}}$, and let

$$\mathbb{C}_r^* = \frac{1}{T}\left(\sum_{k=1}^{K}\sum_{t=1}^{H}\sum_{(s,a)\in L_k} w_{tk}(s,a)\mathrm{Var}R(s,a)\right) \quad ,$$

where L_k is defined in Definition 2. Finally, let $\mathbb{Q}^* := \max_{s,a,t}\big(\mathrm{Var}R(s,a) + \mathrm{Var}_{s'\sim p(\cdot|s,a)}V_{t+1}^*(s')\big)$. Then,

$$\mathbb{C}_r^* + \mathbb{C}^* \lesssim \mathbb{Q}^*$$
$$\mathbb{C}^\pi \lesssim \frac{\mathcal{G}^2}{H}$$
$$\mathbb{C}_r^* \leq \frac{\mathcal{G}^2}{H}$$

Proof. We follow proposition 6 of [Zanette and Brunskill, 2019], and start by substituting $g(p,V)$ into $\mathbb{C}_r^* + \mathbb{C}^*$

$$\begin{aligned}
\mathbb{C}_r^* + \mathbb{C}^* &\lesssim \frac{1}{T}\left(\sum_{k=1}^{K}\sum_{t=1}^{H}\sum_{(s,a)\in L_k} w_{tk}(s,a)\mathrm{Var}R(s,a)\right) \\
&\quad + \frac{1}{T}\sum_{k=1}^{K}\sum_{t=1}^{H}\sum_{s,a} w_{tk}(s,a)\mathrm{Var}_{s'\sim p(\cdot|s,a)}V_{t+1}^*(s') \\
&\leq \frac{1}{T}\left(\sum_{k=1}^{K}\sum_{t=1}^{H}\sum_{(s,a)} w_{tk}(s,a)(\mathrm{Var}R(s,a)) + \mathrm{Var}_{s'\sim p(\cdot|s,a)}V_{t+1}^*(s')\right) \\
&\leq \frac{1}{T}\left(\sum_{k=1}^{K}\sum_{t=1}^{H}\sum_{(s,a)} w_{tk}(s,a)\max_{s,a,t}\{\mathrm{Var}R(s,a)\} + \mathrm{Var}_{s'\sim p(\cdot|s,a)}V_{t+1}^*(s')\right) \\
&= \frac{\mathbb{Q}^*}{T}\left(\sum_{k=1}^{K}\sum_{t=1}^{H}\sum_{(s,a)} w_{tk}(s,a)\right) \\
&= \mathbb{Q}^*
\end{aligned}$$

where the last equality is since $\sum_{(s,a)} w_{tk}(s,a) = 1$ and $T = HK$.

Next, we bound $\mathbb{C}^\pi$:

$$\begin{aligned}
\mathbb{C}^\pi &\lesssim \frac{1}{T}\sum_{k=1}^{K}\sum_{t=1}^{H}\sum_{s,a} w_{tk}(s,a)\mathrm{Var}_{s'\sim p(\cdot|s,a)}V_{t+1}^{\pi_k}(s') \\
&\overset{(1)}{=} \frac{1}{T}\sum_{k=1}^{K}\mathbb{E}\left[\left(\sum_{t=1}^{H} r(s_t^k,a_t^k) - V_1^{\pi_k}(s_1^k)\right)^2 \mid \mathcal{F}_{k-1}\right] \\
&\leq \frac{1}{T}\sum_{k=1}^{K}\mathbb{E}\left[\left(\sum_{t=1}^{H} r(s_t^k,a_t^k)\right)^2 \mid \mathcal{F}_{k-1}\right] \\
&\overset{(2)}{\leq} \frac{1}{T}K\mathcal{G}^2 = \frac{\mathcal{G}^2}{H} \ ,
\end{aligned}$$

where (1) is due to the Law of Total Variance (LTV), which was used in [Azar et al., 2017], and was stated formally in Lemma 15 of [Zanette and Brunskill, 2019]. In (2), we bound the reward in an episode by $\mathcal{G}$.

Finally, the bound on $\mathbb{C}_r^*$ is proven in Lemma 8 of [Zanette and Brunskill, 2019], which concludes this proof.

□

We also prove the following lemma that helps translating bounds that depend on $\mathbb{C}^*$ to bounds that depends on $\mathbb{C}^\pi$. This lemma is equivalent to lemma 14 of [Zanette and Brunskill, 2019], but the prove requires Lemma 27, that was not proved in their paper. This is since they rely on the inequality $\underline{V}_t^{k-1} \leq V^{\pi_k}$, which does not seem to hold.

Lemma 29 (Bound Translation Lemma)**.** *Outside the failure event, it holds that*

$$\begin{aligned}
&\sum_{k=1}^{K}\sum_{t=1}^{H}\sum_{(s,a)\in L_k} w_{tk}(s,a)\frac{g(p,V_{t+1}^*)}{\sqrt{n_{k-1}(s,a)\vee 1}} - \sum_{k=1}^{K}\sum_{t=1}^{H}\sum_{(s,a)\in L_k} w_{tk}(s,a)\frac{g(p,V_{t+1}^{\pi_k})}{\sqrt{n_{k-1}(s,a)\vee 1}} \\
&\qquad = \tilde{\mathcal{O}}\Big(B_v SAH^{\frac{3}{2}}(F+D) + B_v SAH^{\frac{5}{2}}\Big)
\end{aligned}$$

where F, D are defined in Lemma 23.

Proof. We start as in the original Lemma 14 of [Zanette and Brunskill, 2019]:

$$\begin{aligned}
&\sum_{k=1}^{K}\sum_{t=1}^{H}\sum_{(s,a)\in L_k} w_{tk}(s,a)\frac{g(p,V_{t+1}^*)}{\sqrt{n_{k-1}(s,a)\vee 1}} - \sum_{k=1}^{K}\sum_{t=1}^{H}\sum_{(s,a)\in L_k} w_{tk}(s,a)\frac{g(p,V_{t+1}^{\pi_k})}{\sqrt{n_{k-1}(s,a)\vee 1}} \\
&\overset{(1)}{\leq} B_v\sum_{k=1}^{K}\sum_{t=1}^{H}\sum_{(s,a)\in L_k} w_{tk}(s,a)\frac{\left\|V_{t+1}^* - V_{t+1}^{\pi_k}\right\|_{2,p}}{\sqrt{n_{k-1}(s,a)\vee 1}} \\
&\overset{(2)}{\leq} B_v\sqrt{\sum_{k=1}^{K}\sum_{t=1}^{H}\sum_{(s,a)\in L_k}\frac{w_{tk}(s,a)}{n_{k-1}(s,a)\vee 1}}\sqrt{\sum_{k=1}^{K}\sum_{t=1}^{H}\sum_{(s,a)\in L_k} w_{tk}(s,a)\left\|V_{t+1}^* - V_{t+1}^{\pi_k}\right\|_{2,p}^2} \\
&\overset{(3)}{\lesssim} B_v\sqrt{SA}\sqrt{\sum_{k=1}^{K}\sum_{t=1}^{H}\sum_{(s,a)} w_{tk}(s,a)p(\cdot\mid s,a)^T\left(\bar{V}_{t+1}^{k-1} - V_{t+1}^{\pi_k}\right)^2}
\end{aligned}$$

where in (1) we use property 1 of Definition 1, and (2) is due to Cauchy-Schwarz inequality. In (3) we used Lemma 37. Next, we apply Lemma 27 to bound the remaining term by $\tilde{\mathcal{O}}\Big(\sqrt{SAH^3(F+D)^2 + SAH^5}\Big)$ and bound $\sqrt{SAH^3(F+D)^2 + SAH^5} \leq \sqrt{SAH^3(F+D)^2} + \sqrt{SAH^5}$, which yields the desired result □

We are now ready to bound each of the terms of the regret. To bound the first term, we cite Lemma 8 of [Zanette and Brunskill, 2019]:

Lemma 30 (Optimistic Reward Bound)**.** *Outside the failure event, it holds that*

$$\sum_{k=1}^{K}\sum_{t=1}^{H}\sum_{(s,a)\in L_k} w_{tk}(s,a)(\tilde{r}_{k-1}-r)(s_t^k,a_t^k) = \tilde{\mathcal{O}}\Big(\sqrt{\mathbb{C}_r^* SAT}+SA\Big)$$

The next three lemmas correspond to the remaining terms, and follow Lemmas 9,10 and 11 of [Zanette and Brunskill, 2019], with slight modifications:

Lemma 31 (Empirical Transition Bound)**.** *Outside the failure event, it holds that*

$$\sum_{k=1}^{K}\sum_{t=1}^{H}\sum_{(s,a)\in L_k} w_{tk}(s,a)(\hat{p}_{k-1}-p_{k-1})(\cdot\mid s,a)^T V_{t+1}^* = \tilde{\mathcal{O}}\Big(\sqrt{\mathbb{C}^* SAT}+JSA\Big)$$

The following bound also holds:

$$\begin{aligned}&\sum_{k=1}^{K}\sum_{t=1}^{H}\sum_{(s,a)\in L_k} w_{tk}(s,a)(\hat{p}_{k-1}-p_{k-1})(\cdot\mid s,a)^T V_{t+1}^*\\ &= \tilde{\mathcal{O}}\Big(\sqrt{\mathbb{C}^\pi SAT}+JSA+B_v SAH^{\frac{3}{2}}(F+D)+B_v SAH^{\frac{5}{2}}\Big)\end{aligned}$$

where F, D are defined in Lemma 23.

Proof. Similarly to Lemma 9 of [Zanette and Brunskill, 2019], by the definition of ϕ (Definition 1), and outside failure event F^{pv},

$$\begin{aligned}&\sum_{k=1}^{K}\sum_{t=1}^{H}\sum_{(s,a)\in L_k} w_{tk}(s,a)(\hat{p}_{k-1}-p_{k-1})(\cdot\mid s,a)^T V_{t+1}^*\\ &\le \sum_{k=1}^{K}\sum_{t=1}^{H}\sum_{(s,a)\in L_k} w_{tk}(s,a)\left(\frac{g(p,V_{t+1}^*)}{\sqrt{n_{k-1}(s,a)\vee 1}}+\frac{J}{n_{k-1}(s,a)\vee 1}\right) \qquad (24)\\ &\overset{(*)}{\le} \sqrt{\sum_{k=1}^{K}\sum_{t=1}^{H}\sum_{(s,a)\in L_k} w_{tk}(s,a)g(p,V_{t+1}^*)^2}\sqrt{\sum_{k=1}^{K}\sum_{t=1}^{H}\sum_{(s,a)\in L_k}\frac{w_{tk}(s,a)}{n_{k-1}(s,a)\vee 1}}\\ &\quad + J\sum_{k=1}^{K}\sum_{t=1}^{H}\sum_{(s,a)\in L_k}\frac{w_{tk}(s,a)}{n_{k-1}(s,a)\vee 1}\end{aligned}$$

where the last inequality is by Cauchy-Schwarz Inequality. Substituting the definition of $\mathbb{C}^*$, and using Lemma 37, we get

$$\lesssim \sqrt{T\mathbb{C}^*}\sqrt{SA}+JSA\ ,$$

which concludes the first statement of the lemma. To get the second statement, we apply Lemma 29 before inequality $(*)$ and only then use Cauchy-Schwarz Inequality. This creates the additional constant term of $\tilde{\mathcal{O}}\Big(B_v SAH^{\frac{3}{2}}(F+D)+B_v SAH^{\frac{5}{2}}\Big)$. Then, by applying Lemma 37, we get the bound with $\mathbb{C}^\pi$. □

Lemma 32 (Lower Order Term)**.** *Let F, D be the constants defined in Lemma 23. Outside the failure event, it holds that*

$$\sum_{k=1}^{K}\sum_{t=1}^{H}\sum_{(s,a)\in L_k} w_{tk}(s,a)\left|(\hat{p}_{k-1}-p)(\cdot \mid s,a)^T(\bar{V}_{t+1}^{k-1}-V_{t+1}^*)\right| = \tilde{\mathcal{O}}\Big(S^{\frac{3}{2}}AH(F+D+H^{\frac{3}{2}})+S^2AH\Big)$$

Proof. Similarly to Lemma 11 of [Zanette and Brunskill, 2019], by the definition of ϕ (Definition 1), and outside failure event F^{ps},

$$\begin{aligned}
&\sum_{k=1}^{K}\sum_{t=1}^{H}\sum_{(s,a)\in L_k} w_{tk}(s,a)\left|(\hat{p}_{k-1}-p)(\cdot \mid s,a)^T(\bar{V}_{t+1}^{k-1}-V_{t+1}^*)\right| \\
&\lesssim \sum_{k=1}^{K}\sum_{t=1}^{H}\sum_{(s,a)\in L_k} w_{tk}(s,a)\sum_{s'}\sqrt{\frac{p(s'\mid s,a)(1-p(s'\mid s,a))}{n_{k-1}(s,a)\vee 1}}\left|\bar{V}_{t+1}^{k-1}(s')-V_{t+1}^*(s')\right| \\
&\quad+\sum_{k=1}^{K}\sum_{t=1}^{H}\sum_{(s,a)\in L_k} w_{tk}(s,a)\sum_{s'}\frac{\left|\bar{V}_{t+1}^{k-1}(s')-V_{t+1}^*(s')\right|}{n_{k-1}(s,a)\vee 1} \\
&\le \sum_{k=1}^{K}\sum_{t=1}^{H}\sum_{(s,a)\in L_k} w_{tk}(s,a)\sum_{s'}\sqrt{\frac{p(s'\mid s,a)(1-p(s'\mid s,a))}{n_{k-1}(s,a)\vee 1}}\left|\bar{V}_{t+1}^{k-1}(s')-V_{t+1}^*(s')\right| \\
&\quad+\sum_{k=1}^{K}\sum_{t=1}^{H}\sum_{(s,a)\in L_k} w_{tk}(s,a)\frac{HS}{n_{k-1}(s,a)\vee 1}\ ,
\end{aligned}$$

where in the last inequality we used the fact that V_{t+1}^* and $\bar{V}_{t+1}^{k-1}$ are in $[0,H]$, by Lemma 18. Next, using the optimism of the value $\underline{V}_{t+1}^{k-1}\le V_{t+1}^*\le \bar{V}_{t+1}^{k-1}$ (Lemma 22), and since $(1-p)\le 1$ for $p\in[0,1]$, we can bound

$$\begin{aligned}
&\le \sum_{k=1}^{K}\sum_{t=1}^{H}\sum_{(s,a)\in L_k} w_{tk}(s,a)\sum_{s'}\sqrt{\frac{p(s'\mid s,a)}{n_{k-1}(s,a)\vee 1}}\left|\bar{V}_{t+1}^{k-1}(s')-\underline{V}_{t+1}^{k-1}(s')\right| \\
&\quad+HS\sum_{k=1}^{K}\sum_{t=1}^{H}\sum_{(s,a)\in L_k}\frac{w_{tk}(s,a)}{n_{k-1}(s,a)\vee 1} \\
&\overset{(CS)}{\le} \sum_{k=1}^{K}\sum_{t=1}^{H}\sum_{(s,a)\in L_k} w_{tk}(s,a)\sqrt{\frac{Sp(\cdot\mid s,a)^T\big(\bar{V}_{t+1}^{k-1}-\underline{V}_{t+1}^{k-1}\big)^2}{n_{k-1}(s,a)\vee 1}} \\
&\quad+HS\sum_{k=1}^{K}\sum_{t=1}^{H}\sum_{(s,a)\in L_k}\frac{w_{tk}(s,a)}{n_{k-1}(s,a)\vee 1} \\
&\overset{(CS)}{\le} \sqrt{S}\sqrt{\sum_{k=1}^{K}\sum_{t=1}^{H}\sum_{(s,a)\in L_k}\frac{w_{tk}(s,a)}{n_{k-1}(s,a)\vee 1}}\sqrt{\sum_{k=1}^{K}\sum_{t=1}^{H}\sum_{(s,a)\in L_k} w_{tk}(s,a)p(\cdot\mid s,a)^T\big(\bar{V}_{t+1}^{k-1}-\underline{V}_{t+1}^{k-1}\big)^2} \\
&\quad+HS\sum_{k=1}^{K}\sum_{t=1}^{H}\sum_{(s,a)\in L_k}\frac{w_{tk}(s,a)}{n_{k-1}(s,a)\vee 1} \\
&\overset{(*)}{\lesssim} \sqrt{S}\sqrt{SA}\sqrt{SAH^2(F+D)^2+SAH^5}+SH\cdot SA \\
&= \tilde{\mathcal{O}}\Big(S^{\frac{3}{2}}AH(F+D+H^{\frac{3}{2}})+S^2AH\Big)
\end{aligned}$$

(CS) denotes Cauchy-Schwarz. Specifically, the first inequality uses $\sum_{i=1}^n a_ib_i\le\sqrt{n\sum_{i=1}^n a_i^2b_i^2}$. In $(*)$, we used Lemmas 37 and 26. □

Lemma 33 (Optimistic Transition Bound)**.** *Let F, D be the constants defined in Lemma 23. Outside the failure event, it holds that*

$$\sum_{k=1}^{K}\sum_{t=1}^{H}\sum_{(s,a)\in L_k} w_{tk}(s,a)(\tilde{p}_{k-1}-\hat{p}_{k-1})(\cdot\mid s,a)^T\bar{V}_{t+1}^{k-1}$$
$$=\tilde{\mathcal{O}}\left(\sqrt{\mathbb{C}^* SAT}+(J+B_p)SA+B_v SAH\left(F+D+H^{\frac{3}{2}}\right)+B_v SA\sqrt{S^{\frac{1}{2}}H(F+D+H^{\frac{5}{2}})+SH^2}\right)$$

The following bound also holds:

$$\sum_{k=1}^{K}\sum_{t=1}^{H}\sum_{(s,a)\in L_k} w_{tk}(s,a)(\hat{p}_{k-1}-\hat{p}_{k-1})(\cdot\mid s,a)^T V_{t+1}^*$$
$$=\tilde{\mathcal{O}}\left(\sqrt{\mathbb{C}^\pi SAT}+(J+B_p)SA+B_v SAH^{\frac{3}{2}}(F+D+H)+B_v SA\sqrt{S^{\frac{1}{2}}H(F+D+H^{\frac{5}{2}})+SH^2}\right)$$

Proof. Similarly to Lemma 10 of [Zanette and Brunskill, 2019], by the definition of the bonus,

$$\begin{aligned}
&\sum_{k=1}^{K}\sum_{t=1}^{H}\sum_{(s,a)\in L_k} w_{tk}(s,a)(\tilde{p}_{k-1}-\hat{p}_{k-1})(\cdot\mid s,a)^T\bar{V}_{t+1}^{k-1}\\
&=\sum_{k=1}^{K}\sum_{t=1}^{H}\sum_{(s,a)\in L_k} w_{tk}(s,a)b_k^{pv}(s,a)\\
&=\sum_{k=1}^{K}\sum_{t=1}^{H}\sum_{(s,a)\in L_k} w_{tk}(s,a)\left(\phi(\hat{p}_{k-1}(\cdot\mid s,a),\underline{V}_{t+1}^{k-1})+\frac{B_v\|\bar{V}_{t+1}^{k-1}-\underline{V}_{t+1}^{k-1}\|_{2,\hat{p}}}{\sqrt{n_{k-1}(s,a)\vee 1}}+\frac{4J+B_p}{n_{k-1}(s,a)\vee 1}\right)\\
&\le\sum_{k=1}^{K}\sum_{t=1}^{H}\sum_{(s,a)\in L_k} w_{tk}(s,a)\left(\phi(p(\cdot\mid s,a),V^*)+2\frac{B_v\|\bar{V}_{t+1}^{k-1}-\underline{V}_{t+1}^{k-1}\|_{2,\hat{p}}}{\sqrt{n_{k-1}(s,a)\vee 1}}+2\frac{4J+B_p}{n_{k-1}(s,a)\vee 1}\right)\ .
\end{aligned}$$

In the last inequality, we applied Lemma 15, Property(3), and used Equation (13) together with the optimism of the value function, that is $\underline{V}_{t+1}^{k-1}\le V_{t+1}^*\le\bar{V}_{t+1}^{k-1}$ (Lemma 18). Next, we substitute the definition of ϕ (Definition 1), and get

$$\lesssim\sum_{k=1}^{K}\sum_{t=1}^{H}\sum_{(s,a)\in L_k} w_{tk}(s,a)\left(\frac{g(p,V_{t+1}^*)}{n_{k-1}(s,a)\vee 1}+\frac{J+B_p}{n_{k-1}(s,a)\vee 1}\right) \tag{25}$$
$$+\sum_{k=1}^{K}\sum_{t=1}^{H}\sum_{(s,a)\in L_k} w_{tk}(s,a)\frac{B_v\|\bar{V}_{t+1}^{k-1}-\underline{V}_{t+1}^{k-1}\|_{2,\hat{p}}}{\sqrt{n_{k-1}(s,a)\vee 1}}\ . \tag{26}$$

The term in Equation (25) is almost identical to Equation (24) of Lemma 31, and can be similarly bounded by replacing J with $J+B_p$. This yields a bound of either $\tilde{\mathcal{O}}\left(\sqrt{\mathbb{C}^* SAT}+(J+B_p)SA\right)$ or $\tilde{\mathcal{O}}\left(\sqrt{\mathbb{C}^\pi SAT}+(J+B_p)SA+B_v SAH^{\frac{3}{2}}(F+D)+B_v SAH^{\frac{5}{2}}\right)$. We now move to bounding the second term. Notice that

$$\begin{aligned}
\|\bar{V}_{t+1}^{k-1}-\underline{V}_{t+1}^{k-1}\|_{2,\hat{p}}^2&=\hat{p}_{k-1}(\cdot\mid s,a)^T\left(\bar{V}_{t+1}^{k-1}-\underline{V}_{t+1}^{k-1}\right)^2\\
&=p(\cdot\mid s,a)^T\left(\bar{V}_{t+1}^{k-1}-\underline{V}_{t+1}^{k-1}\right)^2+(\hat{p}_{k-1}-p)(\cdot\mid s,a)^T\left(\bar{V}_{t+1}^{k-1}-\underline{V}_{t+1}^{k-1}\right)^2\\
&=\|\bar{V}_{t+1}^{k-1}-\underline{V}_{t+1}^{k-1}\|_{2,p}^2+(\hat{p}_{k-1}-p)(\cdot\mid s,a)^T\left(\bar{V}_{t+1}^{k-1}-\underline{V}_{t+1}^{k-1}\right)^2
\end{aligned} \tag{27}$$

Next, applying Cauchy-Schwartz Inequality on (26), we get

$$
\begin{aligned}
(26) &\leq B_v \sqrt{\sum_{k=1}^{K}\sum_{t=1}^{H}\sum_{(s,a)\in L_k} \frac{w_{tk}(s,a)}{n_{k-1}(s,a)\vee 1}} \sqrt{\sum_{k=1}^{K}\sum_{t=1}^{H}\sum_{(s,a)\in L_k} w_{tk}(s,a)\|\bar{V}_{t+1}^{k-1} - \underline{V}_{t+1}^{k-1}\|_{2,\hat{p}}^2} \\
&\lesssim B_v\sqrt{SA}\sqrt{\sum_{k=1}^{K}\sum_{t=1}^{H}\sum_{(s,a)\in L_k} w_{tk}(s,a)\|\bar{V}_{t+1}^{k-1} - \underline{V}_{t+1}^{k-1}\|_{2,p}^2} \\
&\quad + B_v\sqrt{SA}\sqrt{\sum_{k=1}^{K}\sum_{t=1}^{H}\sum_{(s,a)\in L_k} w_{tk}(s,a)\left|(\hat{p}_{k-1}-p)(\cdot \mid s,a)^T\left(\bar{V}_{t+1}^{k-1} - \underline{V}_{t+1}^{k-1}\right)^2\right|} \ ,
\end{aligned}
$$

where the last inequality is by Lemma 37, substituting (27) and using the inequality $\sqrt{a+b} \leq \sqrt{a}+\sqrt{b}$. The first term can be directly bounded by Lemma 26. The second term can be bounded using Lemma 32 as follows:

$$
\begin{aligned}
&\sum_{k=1}^{K}\sum_{t=1}^{H}\sum_{(s,a)\in L_k} w_{tk}(s,a)\left|(\hat{p}_{k-1}-p)(\cdot \mid s,a)^T\left(\bar{V}_{t+1}^{k-1} - \underline{V}_{t+1}^{k-1}\right)^2\right| \\
&\leq H\sum_{k=1}^{K}\sum_{t=1}^{H}\sum_{(s,a)\in L_k} w_{tk}(s,a)\left|(\hat{p}_{k-1}-p)(\cdot \mid s,a)^T\left(\bar{V}_{t+1}^{k-1} - \underline{V}_{t+1}^{k-1}\right)\right| \\
&= \tilde{\mathcal{O}}\left(S^{\frac{3}{2}}AH(F+D+H^{\frac{5}{2}}) + S^2AH^2\right) \ ,
\end{aligned}
$$

where we trivially bounded the value difference by H at the first inequality (due to Lemma 18) and used Lemma 32 at the second one. Summing both terms yields

$$
\begin{aligned}
(26) &= \tilde{\mathcal{O}}\left(B_v\sqrt{SA}\sqrt{SAH^2(F+D)^2 + SAH^5} + B_v\sqrt{SA}\sqrt{S^{\frac{3}{2}}AH(F+D+H^{\frac{5}{2}}) + S^2AH^2}\right) \\
&= \tilde{\mathcal{O}}\left(B_vSAH\left(F+D+H^{\frac{3}{2}}\right) + B_vSA\sqrt{S^{\frac{1}{2}}H(F+D+H^{\frac{5}{2}}) + SH^2}\right)
\end{aligned}
$$

Combining both bounds on (25) and (26) concludes the proof. □

F General Lemmas

Lemma 34 (On Trajectory Regret to Sum of Decreasing Bounded Processes Regret)**.** *For Algorithm 1 and Algorithm 2 it holds that,*

$$\sum_{k=1}^{K}\sum_{t=1}^{H}\mathbb{E}[\bar{V}_t^{k-1}(s_t^k)-\bar{V}_t^k(s_t^k)\mid\mathcal{F}_{k-1}]=\sum_{t=1}^{H}\sum_{s}\sum_{k=1}^{K}\bar{V}_t^{k-1}(s)-\mathbb{E}[\bar{V}_t^k(s)\mid\mathcal{F}_{k-1}]$$

Proof. The following relations hold.

$$\begin{aligned}
&\sum_{k=1}^{K}\sum_{t=1}^{H}\mathbb{E}[\bar{V}_t^{k-1}(s_t^k)-\bar{V}_t^k(s_t^k)\mid\mathcal{F}_{k-1}] &(28)\\
&=\sum_{k=1}^{K}\sum_{t=1}^{H}\sum_{s}\mathbb{E}[\mathbb{1}\{s=s_t^k\}\bar{V}_t^{k-1}(s)-\mathbb{1}\{s=s_t^k\}\bar{V}_t^k(s)\mid\mathcal{F}_{k-1}]\\
&\overset{(1)}{=}\sum_{t=1}^{H}\sum_{s}\sum_{k=1}^{K}\mathbb{E}[\mathbb{1}\{s=s_t^k\}\bar{V}_t^{k-1}(s)+\mathbb{1}\{s\neq s_t^k\}\bar{V}_t^{k-1}(s)\mid\mathcal{F}_{k-1}]\\
&\qquad\qquad-\mathbb{E}[\mathbb{1}\{s=s_t^k\}\bar{V}_t^{k}(s)+\mathbb{1}\{s\neq s_t^k\}\bar{V}_t^{k-1}(s)\mid\mathcal{F}_{k-1}]\\
&\overset{(2)}{=}\sum_{t=1}^{H}\sum_{s}\sum_{k=1}^{K}\bar{V}_t^{k-1}(s)-\mathbb{E}[\mathbb{1}\{s=s_t^k\}\bar{V}_t^{k}(s)+\mathbb{1}\{s\neq s_t^k\}\bar{V}_t^{k-1}(s)\mid\mathcal{F}_{k-1}]\\
&\overset{(3)}{=}\sum_{t=1}^{H}\sum_{s}\sum_{k=1}^{K}\bar{V}_t^{k-1}(s)-\mathbb{E}[\bar{V}_t^{k}(s)\mid\mathcal{F}_{k-1}]. &(29)
\end{aligned}$$

Relation (1) holds by adding and subtracting $\mathbb{1}\{s\neq s_t^k\}\bar{V}_t^{k-1}(s)$ while using the linearity of expectation. (2) holds since for any event $\mathbb{1}\{A\}+\mathbb{1}\{A^c\}=1$ and since ΔV_t^{k-1} is $\mathcal{F}_{k-1}$ measurable. (3) holds by the definition of the update rule. If state s is visited in the k^{th} episode at time-step t, then both $\bar{V}_t^k(s),\underline{V}_t^k(s)$ are updated. If not, their value remains as in the $k-1$ iteration. □

F.1 The Good Set L_k and Few Lemmas

We introduce that set L_k. The construction is similar to [Dann et al., 2017] and we follow the one formulated in [Zanette and Brunskill, 2019]. The idea is to partition the state-action space at each episode to two sets, the set of state-action pairs that have been visited sufficiently often, and the ones that were not.

Definition 2. *The set L_k is defined as follows.*

$$L_k:=\left\{(s,a)\in\mathcal{S}\times\mathcal{A}:\frac{1}{4}\sum_{j<k}w_j(s,a)\geq H\ln\frac{SAH}{\delta'}+H\right\}$$

where $w_j(s,a):=\sum_{t=1}^{H}w_{tj}(s,a)$

We now state some useful lemmas. See proofs in [Zanette and Brunskill, 2019], Lemma 6, Lemma 7, Lemma 13.

Lemma 35. *Outside the failure event, it holds that if $(s,a)\in L_k$, then*

$$n_{k-1}(s,a)\geq\frac{1}{4}\sum_{j\geq k}w_j(s,a)\ ,$$

which also implies that $n_{k-1}(s,a)\geq H\ln\frac{SAH}{\delta'}+H\geq 1$

Lemma 36. *Outside the failure event, it holds that*

$$\sum_{k=1}^{K}\sum_{t=1}^{H}\sum_{(s,a)\notin L_k}w_{tk}(s,a)\leq\tilde{\mathcal{O}}(SAH).$$

Lemma 37. *Outside the failure event, it holds that*

$$\sum_{k=1}^{K}\sum_{t=1}^{H}\sum_{(s,a)\in L_k}\frac{w_{tk}(s,a)}{n_{k-1}(s,a)} \leq \tilde{\mathcal{O}}(SA).$$

Combining these lemmas we conclude the following one.

Lemma 38. *Outside the failure event, it holds that*

$$\sum_{k=1}^{K}\sum_{t=1}^{H}\mathbb{E}\left[\sqrt{\frac{1}{n_{k-1}(s_t^k,\pi_k(s_t^k))\vee 1}} \mid \mathcal{F}_{k-1}\right] \leq \tilde{\mathcal{O}}(\sqrt{SAT}+SAH)$$

Proof. The following holds relations hold.

$$\begin{aligned}
&\sum_{k=1}^{K}\sum_{t=1}^{H}\mathbb{E}\left[\sqrt{\frac{1}{n_{k-1}(s_t^k,\pi_k(s_t^k))\vee 1}} \mid \mathcal{F}_{k-1}\right] \\
&= \sum_{k=1}^{K}\sum_{t=1}^{H}\sum_{s,a} w_{tk}(s,a)\sqrt{\frac{1}{n_{k-1}(s,a)\vee 1}} \\
&\leq \sum_{k=1}^{K}\sum_{t=1}^{H}\sum_{s,a\in L_k} w_{tk}(s,a)\sqrt{\frac{1}{n_{k-1}(s,a)}} + \sum_{k=1}^{K}\sum_{t=1}^{H}\sum_{s,a\notin L_k} w_{tk}(s,a) \\
&\leq \sum_{k=1}^{K}\sum_{t=1}^{H}\sum_{s,a\in L_k} w_{tk}(s,a)\sqrt{\frac{1}{n_{k-1}(s,a)}} + SAH. \qquad (30)
\end{aligned}$$

The first relation holds by definition. The second relation holds by the following argument. For the first term, if $(s,a)\in L_k$ then by Lemma 35, $n_{k-1}(s,a)\geq 1$, and thus $n_{k-1}(s,a)\vee 1 = n_{k-1}(s,a)$. The second term is bounded by taking the worst case for the fraction, which is $n_{k-1}(s,a)\vee 1 \geq 1$. The third relation holds by Lemma 36.

Consider the first term in (30).

$$\begin{aligned}
&\sum_{k=1}^{K}\sum_{t=1}^{H}\sum_{s,a\in L_k} w_{tk}(s,a)\sqrt{\frac{1}{n_{k-1}(s,a)}} \\
&\leq \sqrt{\sum_{k=1}^{K}\sum_{t=1}^{H}\sum_{s,a\in L_k} w_{tk}(s,a)}\sqrt{\sum_{k=1}^{K}\sum_{t=1}^{H}\sum_{s,a\in L_k}\frac{w_{tk}(s,a)}{n_{k-1}(s,a)}} \\
&\leq \sqrt{\sum_{k=1}^{K}\sum_{t=1}^{H}\sum_{s,a} w_{tk}(s,a)}\sqrt{\sum_{k=1}^{K}\sum_{t=1}^{H}\sum_{s,a\in L_k}\frac{w_{tk}(s,a)}{n_{k-1}(s,a)}} \\
&= \sqrt{T}\sqrt{\sum_{k=1}^{K}\sum_{t=1}^{H}\sum_{s,a}\frac{w_{tk}(s,a)}{n_{k-1}(s,a)}} \lesssim \tilde{\mathcal{O}}(\sqrt{SAT}).
\end{aligned}$$

The first relation holds by Cauchy-Schartz inequality. In the second relation, we replaced the sum in the first term to cover all of the state-action pairs, thus adding positive quantities. The third relation holds since by definition $\sum_{t=1}^{H}\sum_{s,a} w_{tk}(s,a) = H$ and $T = KH$. The last relation holds by Lemma 37.

Combining the result in (30) concludes the proof. □

Lemma 39. *Let $u, v \geq 0$ be some non-negative constants. Outside the failure event,*

$$\sum_{k=1}^{K}\sum_{t=1}^{H}\mathbb{E}\left[\min\left\{\frac{u}{n_{k-1}(s_{t'}^k, a_{t'}^k)\vee 1}, v\right\} \mid \mathcal{F}_{k-1}\right] \leq \tilde{\mathcal{O}}(SAu + SAHv) \ ,$$

and specifically,

$$\sum_{k=1}^{K}\sum_{t=1}^{H}\mathbb{E}\left[\frac{u}{n_{k-1}(s_{t'}^k, a_{t'}^k)\vee 1} \mid \mathcal{F}_{k-1}\right] \leq \tilde{\mathcal{O}}(SAHu) \ .$$

Proof. The proof partially follows [Zanette and Brunskill, 2019], Lemma 12:

$$\begin{aligned}
&\sum_{k=1}^{K}\sum_{t=1}^{H}\mathbb{E}\left[\min\left\{\frac{u}{n_{k-1}(s_{t'}^k, a_{t'}^k)\vee 1}, v\right\} \mid \mathcal{F}_{k-1}\right] \\
&\overset{(1)}{=} \sum_{k=1}^{K}\sum_{t=1}^{H}\sum_{s,a} w_{tk}(s,a)\min\left\{\frac{u}{n_{k-1}(s,a)\vee 1}, v\right\} \\
&\overset{(2)}{\leq} \sum_{k=1}^{K}\sum_{t=1}^{H}\sum_{(s,a)\in L_k} w_{tk}(s,a)\frac{u}{n_{k-1}(s,a)} + v\sum_{k=1}^{K}\sum_{t=1}^{H}\sum_{(s,a)\notin L_k} w_{tk}(s,a) \\
&\overset{(3)}{\lesssim} SAu + SAHv
\end{aligned}$$

(1) is from the definition of $w_{tk}(s,a)$ and the fact that $n_{k-1}(s,a)$ is $\mathcal{F}_{k-1}$ measurable. In (2) we divided the sum into state-actions in and outside L_k. For state-actions in L_k, we bounded the minimum by the first term, and otherwise we bounded it by H^2. Note that for any $(s,a) \in L_k$, $n_{k-1}(s,a) \geq 1$, from Lemma 35. (3) is due to Lemmas 36 and 37.

The second part of the lemma is a direct result of fixing $v = u$.

□